universität wien

DIPLOMARBEIT

Titel der Diplomarbeit

Particle creation via tunneling effect

angestrebter akademischer Grad

Magister der Naturwissenschaften (Mag. rer. nat.)

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Matrikel-Nummer: 0406124
Studienrichtung: A 411 Diplomstudium Physik
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Wien, am 17. 01. 2011

Danksagung

Zunächst möchte ich mich bei jenen Personen bedanken, die mir bei der Erstellung meiner Diplomarbeit geholfen haben.

An erster Stelle bedanke ich mich ganz herzlich bei meinem Betreuer Prof. Helmut Rumpf für seine ausgezeichnete Betreuung. Mit viel Geduld hat er sich meinen Fragen gestellt und mir engagiert bei Problemen geholfen.

Von ganzem Herzen danke ich meinen Eltern Juso und Mevla Kaltak. Sie haben mich während dem Studium sowohl emotional als auch finanziell unterstützt. Solche Eltern kann sich ein Kind nur wünschen.

Meinen Schwestern Edina und Azra Kaltak bin ich sehr verbunden, weil sie stets ein offenes Ohr für mich hatten und mir mit Rat unt Tat zur Seite gestanden sind.

Desweiteren gilt Dank und Anerkennung meinen Freunden und Kollegen. Sie stellten ein Umfeld her, das mir ein erfolgreiches Arbeiten ermöglicht hat.

Zu guter Letzt möchte ich mich bei meiner Freundin Tamara Pinterich bedanken. Ihr fortdauernder Rückhalt hat mich stets dazu bewegt mein Bestes zu geben.

Contents

0. Introduction

Particles play an important role in physics. For instance matter contains atoms which in turn are built up by particles such as leptons and hadrons. Moreover gauge bosons are nowadays understood as the carriers of the fundamental forces of nature. Even for certain phenomena concerning light the particle model is needed.

In modern physics Quantum Field Theory (QFT) seizes on the particle concept to describe relativistic systems. In this context particles come across as excitations of a fundamental quantum state. In absence of matter this special state is the Minkowski vacuum state. Therefore it was surprising when in 1973 Fulling found out that the Minkowski vacuum differs from the natural vacuum of an uniformly accelerated observer [Fulling, 1973].

As is well known there exist two different types of constant accelerated motion, namely constant linear and constant centripetal acceleration. In the first case particle creation appears so that the Minkowski vacuum looks like a heat bath in the accelerated frame. This is the so called Unruh effect. But in the second case the Minkowski vacuum does not contain particles, although accelerated motion is still present. For the latter reason it must still be expected that the rotating observer detects some kind of radiation. Indeed a simple detector model introduced by DeWitt predicts a non-trivial spectrum [Letaw, 1981]. What makes this effect in rotating frames physically so exciting is the opportunity to verify it experimentally [Bell and Leinaas, 1983].

Besides this motivation a recent paper [Parikh and Wilczek, 2000] offered a different approach to the subject. By using the quantum mechanical method of tunneling they described the Hawking radiation which is related closely to the Unruh effect. Thus the quantum field theoretical detector model is no longer needed to understand Unruh radiation.

In the present thesis both the already known detector model as well as the not yet investigated tunneling approach will be used for the description of the Unruh radiation in rotating frames.

1. Rotating frame of reference

In this chapter we introduce the natural frame of reference of an observer sitting in the origin and rotating constantly around an axis. The free particle worldlines of the corresponding system are discussed using Killing vectors.

1.1. Killing vectors and line element

In standard cylindrical coordinates $x = (t, r, \varphi, z)$ the line element of Minkowski spacetime has the form:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\varphi^2 - \mathrm{d}z^2. \quad (c = 1) \tag{1.1}$$

The world line of an observer traveling in a circle with radius $r = r_d$ and constant angular velocity Ω in Minkowski spacetime is given by

$$x^\mu(t) = (t, r_d, \Omega t, 0). \tag{1.2}$$

The tangent vector $\frac{\mathrm{d}x}{\mathrm{d}t}$ is a Killing vector ξ:

$$\xi = \frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\varphi}. \tag{1.3}$$

This is obvious since (1.3) is a linear combination of two Killing vectors $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial\varphi}$. The coordinate transformation

$$\bar{\varphi} = \varphi - \Omega t, \tag{1.4}$$

defines a system $\bar{\Sigma}$ with coordinates $\bar{x} = (t, r, \bar{\varphi}, z)$. According to (1.4) the Killing vector (1.3) is given by

$$\xi = \frac{\partial}{\partial t} \tag{1.5}$$

in the new coordinates. Hence ξ is tangent to the world line of an object at rest in $\bar{\Sigma}$ [Letaw and Pfautsch, 1982]. In addition (1.4) transforms the line element (1.1) (by replacing $\mathrm{d}\varphi \to \mathrm{d}\bar{\varphi} + \Omega\mathrm{d}t$) into:

$$\mathrm{d}s^2 = (1 - \Omega^2 r^2)\mathrm{d}t^2 - 2\Omega r^2\mathrm{d}t\mathrm{d}\bar{\varphi} - \mathrm{d}r^2 - r^2\mathrm{d}\bar{\varphi}^2 - \mathrm{d}z^2. \tag{1.6}$$

From (1.6) we can read off the components $g_{\mu\nu}$ of the metric tensor $g = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ of the rotating frame. This metric is stationary but not static (a global, nonvanishing timelike Killing vector field cannot be found). Special attention must be given to the g_{00} component, which is the norm $g_{\mu\nu}\xi^\mu\xi^\nu$ of the Killing vector (1.5). It has the property that

$$g_{00} = \begin{cases} > 0 & \text{if } 0 < r < \frac{1}{\Omega} \\ = 0 & \text{if } r = \frac{1}{\Omega} \\ < 0 & \text{if } \frac{1}{\Omega} < r < \infty \end{cases}$$

Thus ξ is:

(i) timelike for $r < \frac{1}{\Omega}$,

(ii) spacelike for $r > \frac{1}{\Omega}$,

(iii) null for $r = \frac{1}{\Omega}$.

(iii) is the reason why $\frac{1}{\Omega}$ is sometimes called light cylinder radius.

Because ξ is the four velocity of a stationary object in the rotating frame, $\frac{1}{\Omega}$ marks the stationary limit of the system. Hence a physical object (for instance a patricle detector) resting in the rotating frame is always located inside the light cylinder. Note that there is no event horizon in the rotating system in contrast, for example, to the Rindler frame, the natural frame of an uniformly accelerated observer [Rindler, 2006].

In the following we need also the inverse metric $g^{-1} = g^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}$. It can be obtained by inverting the matrix $(g_{\mu\nu})$. After an elementary calculation [Rizzi and Ruggiero, 2004] one finds that the inverse metric components are

$$(g^{\mu\nu}) = \begin{pmatrix} 1 & 0 & -\Omega & 0 \\ 0 & -1 & 0 & 0 \\ -\Omega & 0 & -\frac{1-\Omega^2 r^2}{r^2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.7}$$

1.2. Particle orbits in the rotating frame

There are several ways to study the world lines of free particles of a system. For instance one could form the geodesic equation and try to solve it. Or one makes use of the symmetries of the metric. Since the the first method is cumbersome[1] we choose the latter approach and follow [Raine and Thomas, 2010].

1.2.1. Constants of motion

Let $X = X^\mu \partial_\mu$ be a Killing vector and $u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}s}$ the four velocity of the particle (i.e. s denotes the proper time). If there are no other forces than gravitation present, the world line of the particle x^μ is a geodesic of the spacetime. Thus u^μ is the tangent of the world line which is paralell transported along the path x^μ. This implies that [Ellis and Hawking, 1994]

$$u^\nu \nabla_\nu u^\mu \equiv u^\nu u^\mu{}_{;\nu} = 0, \tag{1.8}$$

[1]It needs the explicit form of the Christoffel symbols $\Gamma^\mu{}_{\nu\alpha}$.

where ∇_μ is the covariant derivative.

Now we consider the rate of change of the quantity $X^\mu u_\mu$ along the geodesic:

$$u^\nu (X^\mu u_\mu)_{;\nu} = u^\nu u_{\mu;\nu} X^\mu + u^\nu u^\mu X_{\mu;\nu} = 0 + \frac{1}{2} u^\nu u^\mu (X_{\mu;\nu} + X_{\nu;\mu}) = 0$$

The last term vanishes due to the defining property of the Killing vector X,

$$\mathcal{L}_X g_{\mu\nu} = X_{\mu;\nu} + X_{\nu;\mu} = 0.$$

Therefore $X^\mu u_\mu$ is a constant of motion. In the following we investigate $X^\mu u_\mu$ for all Killing vectors provided by the rotating system.

1.2.2. Conserved energy

Energy is the constant of motion associated with time translation symmetry [Scheck, 2002], thus the Killing vector (1.5) defines the conserved (specific) energy $\bar{e}$ as measured by the rotating observer of the free particle. Using the abbreviation $\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}s}$ we have

$$\bar{e} := g_{\mu\nu} \xi^\mu u^\nu = g_{00} u^0 + g_{02} u^2 = (1 - \Omega^2 r^2)\dot{t} - \Omega r^2 \dot{\varphi} = constant$$

and because of $u_0 = g_{0\nu} u^\nu$ we obtain

$$u_0 = \bar{e} = \dot{t} - \Omega^2 r^2 \dot{t} - \Omega r^2 \dot{\varphi}. \tag{1.9}$$

1.2.3. Conserved angular momentum

Due to the fact that the vector $\frac{\partial}{\partial \varphi}$ is a Killing vector the (specific) angular momentum $\bar{l}_z$ with respect to the z-axis is constant. We obtain

$$-\bar{l}_z := g_{\nu 2} u^\nu 1 = g_{02} u^0 + g_{22} u^2 = -\Omega r^2 \dot{t} - r^2 \dot{\varphi}$$

$$\Rightarrow u_2 = -\bar{l}_z = -r^2 (\Omega \dot{t} + \dot{\varphi}) \tag{1.10}$$

1.2.4. Conserved momentum

Similarly the Killing vector $\frac{\partial}{\partial z}$ implies that the motion in z- direction is conserved. Hence

$$u_3 = -\bar{p}_z = -\dot{z}, \tag{1.11}$$

where $-\bar{p}_z := g_{\nu 3} u^\nu$.

1.2.5. Radial motion

To finally find the radial component of the four velocity u_1 we make use of the line element (1.6) and divide by $\mathrm{d}s$. This yields

$$1 = (1-\Omega^2 r^2)\dot{t}^2 - 2\Omega r^2 \dot{t}\dot{\varphi} - \dot{r}^2 - r^2\dot{\varphi}^2 - \dot{z}^2 \Rightarrow \dot{r}^2 = \dot{t}^2 - \dot{z}^2 - 1 - r^2(\Omega\dot{t}+\dot{\varphi})^2. \tag{1.12}$$

Combining (1.9) and (1.10) we conclude that $\dot{t} = \bar{e} + \Omega\bar{l}_z$ and $r^2(\Omega\dot{t}+\dot{\varphi})^2 = \frac{\bar{l}_z^2}{r^2}$. Defining the quantity

$$\bar{q}^2 := (\bar{e}+\Omega\bar{l}_z)^2 - \bar{p}_z^2 - 1, \tag{1.13}$$

inserting (1.11) into (1.12), the last missing component of the four velocity is:

$$u_1 = g_{11}\dot{r} = \mp\sqrt{\bar{q}^2 - \frac{\bar{l}_z^2}{r^2}}. \tag{1.14}$$

The '$-$' stands for a radially outgoing particle[2] and the positive sign for an incoming one. In the following we consider the negative case only by droping the $+$ sign. Further note that motion with

$$\bar{q}^2 r^2 < \bar{l}_z^2, \tag{1.15}$$

is forbidden and therefore

$$r_L := \frac{|\,\bar{l}_z\,|}{\bar{q}} \tag{1.16}$$

marks the classical turning point of the radial motion. This will be confirmed in chapter 2, where we consider the quantized system.

Summarizing the results above we have for the four velocity the following expression

$$u_\mu = (\bar{e}, -\sqrt{\bar{q}^2 - \frac{\bar{l}_z^2}{r^2}}, -\bar{l}_z, -\bar{p}_z). \tag{1.17}$$

1.2.6. Four momentum of the free particle

The covariant components of the four momentum p_μ of the particle are related to those of the four velocity by $p_\mu = m_0 u_\mu$. Where m_0 denotes the rest mass of the particle. Hence using (1.17) we have

$$p_\mu = (m_0\bar{e}, -m_0\sqrt{\bar{q}^2 - \frac{\bar{l}_z^2}{r^2}}, -m_0\bar{l}_z, -m_0\bar{p}_z).$$

The mass shell condition $p_\mu p^\mu = m_0^2$ follows from $u_\mu u^\mu = 1$.

The zero component of the four momentum has dimension of energy. Because m_0 has dimension of energy in natural units, $\bar{e}$ is dimensionless. Thus $\bar{e}$ has the meaning of the

[2] Because for outgoing motion we have $u^1 = \dot{r} > 0$, therefore $u_1 = g_{11}u^1 < 0$.

specific energy (energy per rest mass) of the particle. Analogous interpretations hold for $\bar{l}_z$, $\bar{p}_z$ and $\bar{q}$.

We introduce the symbols $\bar{E}$, $\bar{L}_z$, $\bar{P}_z$, $\bar{Q}$ to denote the corresponding observables for arbitary values of the rest mass. Due to

$$\bar{Q}^2 = p_1^2 + \frac{p_2^2}{r^2} \tag{1.18}$$

we can regard $\bar{Q}$ as the absolute value of the momentum in the $(r, \bar{\varphi})$-plane. Thus physically the reality condition (1.15) tells us that the radial coordinate of the particle is always larger than its angular momentum divided by its momentum in the $(r, \bar{\varphi})$-plane.

1.2.7. Geodesics of the rotating system

Particles travel along geodesics through spacetime. Geometrically the four velocity of a particle represents the tangent to the geodesic [Ellis and Hawking, 1994]. Hence the corresponding integral curves of u^μ are the geodesics of the system. By raising the indices with the inverse metric (1.7) according to $\dot{x}^\mu = u^\mu = g^{\mu\nu} u_\nu$ we obtain:

$$\dot{t} = \frac{1}{m_0}(\bar{E} + \Omega \bar{L}_z) \tag{1.19}$$

$$\dot{r} = \frac{1}{m_0}\sqrt{\bar{Q}^2 - \frac{\bar{L}_z^2}{r^2}} \tag{1.20}$$

$$\dot{\bar{\varphi}} = \frac{1}{m_0}\left(\frac{\bar{L}_z}{r^2} - \Omega(\bar{E} + \Omega \bar{L}_z)\right) \tag{1.21}$$

$$\dot{z} = \frac{\bar{P}_z}{m_0}. \tag{1.22}$$

The integration of this system of ordinary differential equations with the initial values $t(0) = 0$, $r(0) = r_L$, $\bar{\varphi}(0) = \bar{\varphi}_0$, $z(0) = 0$ gives

$$t(s) = \frac{s}{m_0}(\bar{E} + \Omega \bar{L}_z) \tag{1.23}$$

$$r(s) = \sqrt{\frac{\bar{L}_z^2}{\bar{Q}^2} + \frac{\bar{Q}^2}{m_0^2} s^2} \tag{1.24}$$

$$\bar{\varphi}(s) = \bar{\varphi}_0 - \frac{\Omega}{m_0}(\bar{E} + \Omega \bar{L}_z)s + \arctan \frac{\bar{Q}^2 s}{m_0 \bar{L}_z} \tag{1.25}$$

$$z(s) = \frac{\bar{P}_z s}{m_0}. \tag{1.26}$$

Fig. 1.1 shows a typical trajectory of a free particle in the $(r, \bar{\varphi})$-plane, where all parameters are set equal to one[3], except $\bar{P}_z = 0$, $\Omega = \frac{\pi}{2}\frac{eV}{\hbar} = (2.4 \times 10^{15} s^{-1})$ and $\bar{\varphi}_0 = \frac{\pi}{4}$. The dashed green curve represents the radially incoming and the blue curve indicates the radially

[3] As measured in natural units ($c = 1 = \hbar$). In SI units we have $\bar{E} = 1eV = 1.6 \times 10^{-19} \frac{kgm^2}{s^2}$, $m_0 = 1\frac{eV}{c^2} = 1.8 \times 10^{-36} kg$, $\bar{L}_z = 1\hbar = 1.1 \times 10^{-34} \frac{kgm^2}{s}$, $\bar{Q} = 1\frac{eV}{c} = 5.3 \times 10^{-28} \frac{kgm}{s}$.

outgoing particle. The region inside the red circle is classically forbidden, i.e. it is the region where $\bar{Q}^2 r^2 - \bar{L}_z^2 < 0$. As $\Omega \to 0$ the particle path tends to a straight line (blue $\to$ solid gray, dashed green $\to$ dashed gray). This is the corresponding free particle trajectory in the inertial frame of reference.

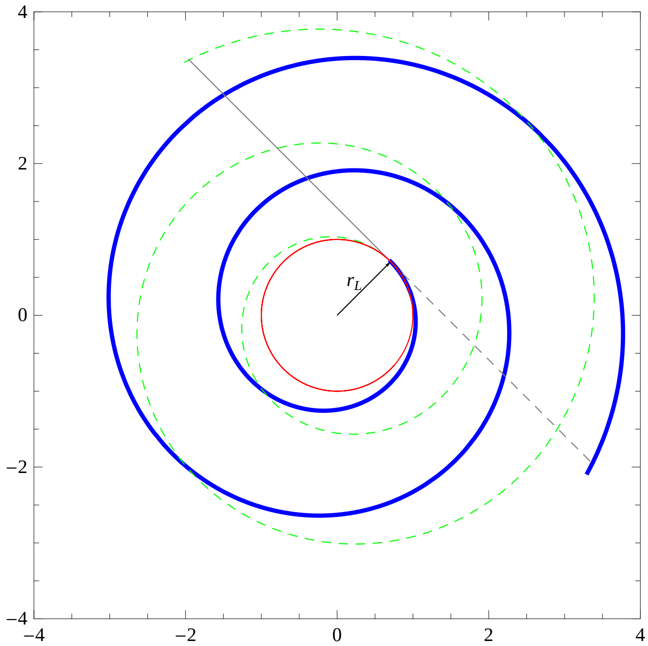

Figure 1.1.: Particle trajectory in $(r, \bar{\varphi})$-plane

1.3. Energy defined by rotating observer

It is important to point out that the energy $\bar{E}$ defined by the rotating observer can become negative in contrast to the energy of a free particle in an inertial frame ($\bar{E}$ is not bounded from below). This can be seen by using $p_\mu p^\mu = m_0^2$ to express $p_0 = \bar{E}$ in terms of the spatial momenta p_1, p_2, p_3:

$$\bar{E} = \sqrt{m_0^2 + p_1^2 + \frac{p_2^2}{r^2} + p_3^2} + \Omega p_2 = \sqrt{m_0^2 + \bar{Q}^2 + \bar{P}_z^2} - \Omega \bar{L}_z. \tag{1.27}$$

For Ω positive and sufficiently large angular momentum $\bar{L}_z$, $\bar{E}$ becomes negative. The corresponding particles satisfy

$$\bar{E} = \sqrt{m_0^2 + p_1^2 + \frac{\bar{L}_z^2}{r^2} + \bar{P}_z^2} - \Omega \bar{L}_z < 0 \Longleftrightarrow m_0^2 + p_1^2 + \frac{\bar{L}_z^2}{r^2} + \bar{P}_z^2 < \bar{L}_z^2 \Omega^2.$$

Because the l.h.s. of the last expression is always larger than $\bar{L}_z^2/r^2$, we have

$$\bar{L}_z^2(\frac{1}{r^2} - \Omega^2) < 0,\ \bar{L}_z^2 > 0 \Longrightarrow \frac{1}{r^2} < \Omega^2 \Longleftrightarrow \underline{r > \frac{1}{\Omega}}.$$

Thus all particles with negative energy are located outside the light cylinder. They are not allowed to exist inside the stationary limit $1/\Omega$. This is in contrast to positive energy particles[4] with $\bar{E} > 0$ which can be located inside $r < 1/\Omega$.

Note that
$$\lim_{\Omega \to 0} \bar{E} = \sqrt{m_0 + p_1^2 + \frac{p_2^2}{r^2} + p_3^2}$$
reproduces the relativistic Hamilton function of the free particle in the inertial frame [Scheck, 2002].

[4] Another way to see this is to consider the turning point r_L. For large energies, r_L is shifted arbitrarily close to 0.

2. Quantum mechanics of free particles in rotating frame

In the previous chapter we studied the classical free point particle in the rotating frame. In the following we want to investigate the corresponding quantized system and focus on relativistic quantum mechanics of free particles. Although a more profound theory of relativistic particles needs the concept of field (which is discussed in chapter 3), the heuristic approach made in this chapter gives valuable results.

2.1. Klein-Gordon equation and Inner product

In Quantum mechanics the wavefunction ψ carries the entire physical information. ψ is the solution of a wave equation. For relativistic particles with spin 0 (which we consider in the following) this is the Klein-Gordon equation (KGE). In order to find the KGE of the rotating system we make use of the first quantization rules [Bjorken and Drell, 1964] (A different derivation is performed in B.2).

We start with rewriting the relativistic energy momentum relation of the rotating system (1.27) into

$$\bar{E}^2 + \bar{L}_z^2\Omega^2 + \Omega\bar{L}_z\bar{E} + \Omega\bar{E}\bar{L}_z - (\bar{Q}^2 + \bar{P}_z^2) - m_0^2 = 0. \tag{2.1}$$

This must hold for the eigenvalues of following operators

- Energy operator: $\hat{\bar{H}} = i\frac{\partial}{\partial t}$
- Angular momenutm operator: $\hat{\bar{L}}_z = -i\frac{\partial}{\partial\bar{\varphi}}$
- Square of spatial momentum operator[1] $\hat{\bar{Q}}^2 + \hat{\bar{P}}_z^2 = -\frac{1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2}{\partial\bar{\varphi}^2} - \frac{\partial^2}{\partial z^2}$.

The conservation of angular momentum translates in operator language into $\left[\hat{\bar{H}}, \hat{\bar{L}}_z\right] = 0$. Taking this commutation relation and the first quantization rules into account the Klein-Gordon equation in the rotating frame reads

$$-\frac{\partial^2\psi}{\partial t^2} + 2\Omega\frac{\partial^2\psi}{\partial t\partial\bar{\varphi}} - \Omega^2\frac{\partial^2\psi}{\partial\bar{\varphi}^2} + \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\bar{\varphi}^2} + \frac{\partial^2\psi}{\partial z^2} - m_0^2\psi = 0. \tag{2.2}$$

[1] This is essentially the negative Laplacian operator $-\Delta$ according to the first quantization rules. In cylindrical coordinates $(r, \bar{\varphi}, z)$ we have $\Delta = \frac{1}{r}\partial_r + \partial_r^2 + \frac{1}{r^2}\partial_{\bar{\varphi}}^2 + \partial_z^2$ [Bronstein et al., 2008].

The solutions of (2.2) need to be normalized. The usual non-relativistic scalar product

$$\langle\psi_1|\,\psi_2\rangle = \int \mathrm{d}^3x\psi_1^*(x)\psi_2^*(x) \tag{2.3}$$

is not conserved for solutions of the KGE. An appropriate covariant inner product is given by [Parker and Toms, 2009]

$$(\psi_1,\psi_2) := i\int_{\Sigma_t} \mathrm{d}\Sigma^\mu \psi_1^* \overleftrightarrow{\partial_\mu}\psi_2, \tag{2.4}$$

where

$$f_1\overleftrightarrow{\partial_\mu}f_2 := f_1\left(\frac{\partial f_2}{\partial x^\mu}\right) - \left(\frac{\partial f_1}{\partial x^\mu}\right)f_2.$$

The integration is performed over a spacelike hypersurface Σ_t. For simplicity we choose the $\{t=0\}$ surface, which is the usual choice in every inertial frame [Scheck, 2007]. The corresponding vector valued hypersurface element $\mathrm{d}\Sigma^\mu$ takes the form

$$\mathrm{d}\Sigma^\mu = \mathrm{d}^3x\sqrt{-g}g^{0\mu} = \mathrm{d}z\mathrm{d}\bar{\varphi}\mathrm{d}rrg^{0\mu}.$$

This implies that the inner product (2.4) in the rotating frame reads

$$(\psi_1,\psi_2) := i\int\limits_{-\infty}^{\infty}\mathrm{d}z\int\limits_0^{2\pi}\mathrm{d}\bar{\varphi}\int\limits_0^{\infty}\mathrm{d}rr\psi_1^*\left(\frac{\overleftrightarrow{\partial}}{\partial t} - \Omega\frac{\overleftrightarrow{\partial}}{\partial\bar{\varphi}}\right)\psi_2. \tag{2.5}$$

(2.4) is sometimes called charge form of the KGE. The reason for this is that in the case of a charged field with coupling e the conserved 4-vector

$$j^\mu := ie\psi^*\overleftrightarrow{\partial^\mu}\psi$$

is interpreted as the electric 4-current density rather than the probability current density[2] [Bjorken and Drell, 1964].

2.2. Wavefunction in the rotating frame

The general separating solution of (2.2) is [Bronstein et al., 2008]

$$\psi(t,r,\bar{\varphi},z) = \bar{a}J_{\bar{m}}(\bar{q}r)e^{-i\bar{\omega}t+i\bar{m}\bar{\varphi}+i\bar{k}_zz} + \bar{b}Y_{\bar{m}}(\bar{q}r)e^{-i\bar{\omega}t+i\bar{m}\bar{\varphi}+i\bar{k}_z}, \tag{2.6}$$

where $J_{\bar{m}}$ denotes the Bessel function of first kind and $Y_{\bar{m}}$ the Neumann function of $\bar{m}$th order. The constants $\bar{a},\bar{b}$ are needed for normalization and will be fixed later on. In general they depend on the eigenvalues $\bar{q}$, $\bar{m}$, $\bar{k}_z$, $\bar{\omega}$ of the operators $\hat{\bar{Q}}$, $\hat{\bar{L}}_z$, $\hat{\bar{P}}_z$, $\hat{\bar{H}}$. By inserting the solution (2.6) back into the Klein-Gordon equation we obtain the dispersion relation in the

[2]Note that in non relativistic QM $\psi^*\overleftrightarrow{\nabla}\psi$ is the probability current density with $|\psi|^2$ the corresponding probability density.

rotating frame

$$\bar{\omega} = \pm\sqrt{m_0^2 + \bar{q}^2 + \bar{k}_z^2} - \Omega\bar{m}. \tag{2.7}$$

It is convenient to choose the positive root, so that for $\Omega \to 0$ (inertial frame) $\bar{\omega}$ is always positive. Nevertheless the energy[3] still becomes negative for sufficiently large angular momentum $\bar{m}$, as mentioned at the end of Chapter 1. In order to obtain a realistic wave function we need to impose boundary conditions.

We are searching for a solution, which is regular at the origin. Hence the coefficient $\bar{b}$ must vanish, due to the divergence behaviour of the Neumann functions at $r = 0$ [Bronstein et al., 2008]. Additionaly we demand 2π-periodicity of ψ (, i.e. $\psi|_{\bar{\varphi}=0} = \psi|_{\bar{\varphi}=2\pi}$). Thus $\bar{m}$ must be an integer. This quantizes the angular momentum $\bar{L}_z$ of the particle. We do not consider spatial boundaries at this point (this will be done in 4.6), which means that $\bar{q}$ and $\bar{k}_z$ are continuous. Consequently the unnormalized free particle wavefunction is

$$\psi_{\vec{\bar{q}}}(t, r, \bar{\varphi}, z) = \bar{a} J_{\bar{m}}(\bar{q}r) e^{-i\bar{\omega}t + i\bar{m}\bar{\varphi} + i\bar{k}_z z}, \quad \vec{\bar{q}} := (\bar{q}, \bar{m}, \bar{k}_z). \tag{2.8}$$

(2.8) is not square integrable and therefore cannot be normalized with respect to (2.5) in the usual way[4]. It must be treated as a distribution [Mukhanov and Winitzki, 2007]. Because $\bar{m}$ is an integer the orthogonality relation for the wavefunction is

$$\left(\psi_{\vec{\bar{q}}}, \psi_{\vec{\bar{q}}'}\right) = \delta_{\bar{m}\bar{m}'}\delta(\bar{k}_z - \bar{k}_z')\frac{\delta(\bar{q} - \bar{q}')}{\bar{q}}. \tag{2.9}$$

In the following we use (2.9) to find the normalization constant $\bar{a}$.

Inserting (2.8) into the r.h.s. gives:

$$\begin{aligned}
\left(\psi_{\vec{\bar{q}}}, \psi_{\vec{\bar{q}}'}\right) &= \bar{a}^*\bar{a}'\left(\bar{\omega}' + \Omega\bar{m}' + \bar{\omega} + \Omega\bar{m}\right) \underbrace{\int_{-\infty}^{\infty} \mathrm{d}z e^{iz(\bar{k}_z' - \bar{k}_z)}}_{2\pi\delta(\bar{k}_z' - \bar{k}_z)} \underbrace{\int_0^{2\pi} \mathrm{d}\bar{\varphi} e^{i\bar{\varphi}(\bar{m}' - \bar{m})}}_{2\pi\delta_{\bar{m}'\bar{m}}} \int_0^{\infty} \mathrm{d}r r J_{\bar{m}'}(\bar{q}'r) J_{\bar{m}}(\bar{q}r) \\
&= \bar{a}^*\bar{a}'\left(\bar{\omega}' + \Omega\bar{m} + \bar{\omega} + \Omega\bar{m}\right)(2\pi)^2\delta(\bar{k}_z' - \bar{k}_z)\delta_{\bar{m}'\bar{m}} \underbrace{\int_0^{\infty} \mathrm{d}r r J_{\bar{m}}(\bar{q}'r) J_{\bar{m}}(\bar{q}r)}_{\frac{\delta(\bar{q}' - \bar{q})}{\bar{q}}}
\end{aligned}$$

In the last step the orthogonality relation of the Bessel functions was used (see B.5). This is allowed since the Kronecker symbol $\delta_{\bar{m}'\bar{m}}$ turns the Bessel functions into the same order of $\bar{m}$. The presence of the delta functions allows us to set $\bar{a}' = \bar{a}$ and $\bar{\omega}' = \bar{\omega}$, so that after comparison with the r.h.s. of (2.9), we have

$$\bar{a} = \frac{1}{2\pi}\frac{1}{\sqrt{2(\bar{\omega} + \bar{m}\Omega)}}. \tag{2.10}$$

[3]In natural units we have the trivial relation $\bar{E} = \bar{\omega}$ between the energy and the frequency. The same holds for angular momentum $\bar{L}_z = \bar{m}$, momentum in $(r, \bar{\varphi})$-plane $\bar{Q} = \bar{q}$ and 3-component of momentum $\bar{P}_z = \bar{k}_z$.

[4]It is the analogue of a plane wave in Cartesian coordinates. If one wants to study finite norm solutions, wave packets of the form $\Psi = \sum_{\bar{m}=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\bar{k}_z \int_0^{\infty} \mathrm{d}\bar{q}\bar{q} f(\vec{\bar{q}})\psi_{\vec{\bar{q}}}(\bar{x})$ must be considered. A Gaussian-like function f would yield to a Gaussian wave packet for instance.

Therefore the normalized wave function of the free particle is

$$\psi_{\bar{\vec{q}}}(t, r, \bar{\varphi}, z) = \frac{1}{2\pi} \frac{1}{\sqrt{2(\bar{\omega} + \bar{m}\Omega)}} J_{\bar{m}}(\bar{q}r) e^{-i\bar{\omega}t + i\bar{m}\bar{\varphi} + i\bar{k}_z z}, \tag{2.11}$$

where

$$\bar{m} \in \mathbb{Z},\ 0 < \bar{q} < \infty,\ -\infty < \bar{k}_z < \infty,\ \bar{\omega} = \sqrt{m_0^2 + \bar{q}^2 + \bar{k}_z^2} - \Omega\bar{m}.$$

2.2.1. Validity of the single particle interpretation

Conservation of relativistic energy-momentum allows the particle number of a relativistic system to be non-constant. This phenomenon (in particular pair creation) must be taken into account by the theory describing relativistic quantized systems[5]. Therefore it is clear that a single particle interpretation of the Klein-Gordon equation cannot be completely consistent. This is shown for instance by the Klein Paradox, where ambiguities appear if one tries to interpret ψ as a single-particle wavefunction [Klein, 1929]. However one can show that these ambiguities become negligible if the characteristic scale of the system Δ becomes very large compared to the Compton wavelength $\lambda_c = \frac{1}{m_0}$ (e.g. for a KG particle penetrating a step potential Δ is given by the penetration depth of the wavefunction in the forbidden zone). Indeed if this scale is large (with respect to λ_c) Klein-Gordon solutions can be considered as single-particle wavefunctions [Wachter, 2005]. Thus we expect that (2.11) describes the wavefunction of a single particle[6] in the rotating frame for large rest masses m_0 compared to Δ^{-1}.

2.3. The possibility of tunneling

In 1.3 we found that the interior of the light cylinder is classically forbidden for negative energy particles. Nevertheless, due to the existence of tunneling phenomena in quantum theory, particles with negative energy will be found inside the light cylinder with a certain probability. Based on this fact, it is conceivable that a particle with negative energy $\bar{\omega} < 0$ and certain quantum numbers $(\bar{q}, \bar{m}, \bar{k}_z)$ may tunnel from $r > \frac{1}{\Omega}$ (more precisely from $r = r_L$, cf. (1.16)) to $r = r_d < \frac{1}{\Omega}$. This tunneling process should be closely related to the excitation of the following quantum system: We assume that the system (called detector in the following and located at $r = r_d$) has two energy eigenstates $|E_0\rangle, |E_1\rangle$ with associated energy eigenvalues E_0, E_1, $(E_1 > E_0)$. The detector is coupled to the field Φ (consisting of free particle wavefunctions of the form (2.11) (cf. chapter 3)), via a monopole moment μ. The corresponding interaction Hamiltonian is

$$\hat{H}_I = \lambda\hat{\mu} \otimes \hat{\Phi}, \tag{2.12}$$

[5]Even if there is not enough energy present ($E < 2m_0$), many-particle states appear in second-order pertubation theory as intermediate states. They can be thought of as states appearing for very short time according to the uncertainty relation $\Delta E \Delta t > \hbar$.

[6]In QM one should work with wave packets rather than plane waves in order to describe particles. A KG wave packet describes a single particle if the spatial extension Δ of the wave packet is much larger than the compton wavelenght λ_c of the particle. Nevertheless for our considerations it is sufficient to work with (2.11).

where λ is a coupling constant, $\hat{\mu}$ the monopole moment operator of the detector and $\hat{\Phi}$ the field operator of the field Φ (cf. chapter 4). The tensor product $\otimes$ indicates that $\hat{H}_I$ is the Hamiltonian of a system consisting of two subsystems (field and detector).

If the detector is initially in its ground state $|E_0\rangle$ there are two possibilities for it to get excited:

- by absorbing a field quantum (particle) of positive energy $\bar{\omega} = \Delta E$ where $\Delta E := E_1 - E_0 > 0$,
- or by emitting a particle with negative energy[7] $\bar{\omega} = -\Delta E < 0$.

We assume that the field is initially in its ground state too. Hence we focus on the emission of negative energy particles. This process is possible if those particles are located at $r_d < \frac{1}{\Omega}$. Therefore we expect that the excitation probability of the detector is proportional to the probability of a particle with energy $\bar{\omega} = -\Delta E$ to tunnel from $r = r_L$ to $r = r_d$.

In order to find this tunneling probability we need to associate a probability measure in position space with a KG wavefunction. As the probability density we take the charge density $j^0 = i\psi^*_{\vec{q}}\overleftrightarrow{\partial^0}\psi_{\vec{q}}$. For the wavefunction (2.11) we get

$$j^0 = ig^{0\mu}\psi^*_{\vec{q}}\overleftrightarrow{\partial_\mu}\psi_{\vec{q}} = 2(\bar{\omega} + \bar{m}\Omega)\left|\psi_{\vec{q}}\right|^2 = \frac{J^2_{\bar{m}}(\bar{q}r)}{4\pi^2}. \tag{2.13}$$

According to our choice the probability P for the particle to tunnel from r_L to r_d is the fraction of the corresponding charge densities at these points [Schiff, 1968]. Writing $\bar{q}$ in terms of energy, angular and spatial momentum (1.13), the probability for finding a particle with negative energy $\bar{\omega} = -\Delta E$ at r_d in the rotating frame is given by

$$j^0(r_d) = \frac{1}{4\pi^2}J^2_{\bar{m}}\left(r_d\sqrt{(-\Delta E + \bar{m}\Omega)^2 - \bar{k}_z^2 - m_0^2}\right). \tag{2.14}$$

The same particle is located at r_L with the probability

$$j^0(r_L) = \frac{J^2_{\bar{m}}(\bar{m})}{4\pi^2}. \tag{2.15}$$

Hence the tunneling probability is given by

$$P = \frac{J^2_{\bar{m}}\left(r_d\sqrt{(-\Delta E + \bar{m}\Omega)^2 - \bar{k}_z^2 - m_0^2}\right)}{J^2_{\bar{m}}(\bar{m})}. \tag{2.16}$$

The formula above describes the tunneling of a particle with fixed values of $\bar{m}$ and $\bar{k}_z$. Due to the degeneracy of the energy eigenstates there exist other combinations of $\bar{m}$ and $\bar{k}_z$ (and $\bar{q}$) which have the same energy $\bar{\omega}$. Since we are interested in the total probability we need to take all states with energy $\bar{\omega} = -\Delta E$ into account. In order to find the allowed combinations of $\bar{m}$ and $\bar{k}_z$ we make use of the results found in 1.3.

For $\bar{E} = -\Delta E$ the relativistic energy-momentum relation (1.27) can be rewritten as ($\bar{L}_z = \bar{m}$, $\bar{P}_z = \bar{k}_z, \bar{Q} = \bar{q}$):

$$(-\Delta E - \bar{m}\Omega)^2 - \bar{k}_z^2 - m_0^2 = \bar{q}^2.$$

[7]This is possible due to the existence of negative energy particles in the rotating frame.

Because the r.h.s. is always positive we have

$$-\sqrt{(-\Delta E+\bar{m}\Omega)^2-m_0^2}<\bar{k}_z<\sqrt{(-\Delta E+\bar{m}\Omega)^2-m_0^2}. \tag{2.17}$$

Due to $-\Delta E+\bar{m}\Omega>0$, which follows from the choice of the positive root in (1.27), we conclude that $\bar{k}_z\in\mathbb{R}$ if

$$(-\Delta E+\bar{m}\Omega)^2>m_0 \Longleftrightarrow \bar{m}>\frac{\Delta E+m_0}{\Omega}. \tag{2.18}$$

Taking (2.17) and (2.18) into account and using the abbreviation

$$\alpha_{\bar{m}}:=\sqrt{(-\Delta E+\bar{m}\Omega)^2-m_0^2},$$

the total tunneling probability, denoted by Γ, is given by

$$\Gamma:=\sum_{\bar{m}=\lceil\frac{\Delta E+m_0}{\Omega}\rceil}^{\infty}\frac{1}{J_{\bar{m}}^2(\bar{m})}\int_{-\alpha_{\bar{m}}}^{\alpha_{\bar{m}}}\mathrm{d}\bar{k}_z J_{\bar{m}}^2\left(r_d\sqrt{(-\Delta E+\bar{m}\Omega)^2-\bar{k}_z^2-m_0^2}\right). \tag{2.19}$$

The symbol $\lceil\rceil$ indicates the ceiling function[8]. It is possible to evaluate the integral analytically:

Note that the integrand is symmetric in $\bar{k}_z$. Hence we are allowed to replace the integral by $2\int_0^{\alpha_{\bar{m}}}$. The variable substitution $q:=\sqrt{\alpha_{\bar{m}}^2-\bar{k}_z^2}$ turns the integration measure $\mathrm{d}\bar{k}_z$ into $-\mathrm{d}q\frac{q}{\sqrt{\alpha_{\bar{m}}^2-q^2}}$ and the integration limits into $\int_{\alpha_{\bar{m}}}^0$ so that we obtain for the integral in (2.19)

$$-\int_{\alpha_{\bar{m}}}^0\mathrm{d}q\frac{qJ_{\bar{m}}^2(qr)}{\sqrt{\alpha_{\bar{m}}^2-q^2}}.$$

This integral can be written in terms of the hypergeometric function ${}_1F_2$ (cf. [Prudnikov et al., 1983], p. 212):

$$\frac{\alpha_{\bar{m}}^{2\bar{m}+1}r_d^{2\bar{m}}}{(2\bar{m}+1)!}\ {}_1F_2\left(\frac{2\bar{m}+1}{2};\frac{2\bar{m}+3}{2},2\bar{m}+1;-\alpha_{\bar{m}}^2r_d^2\right).$$

The final expression for the tunneling probability Γ is therefore

$$\Gamma(\Delta E)=\sum_{\bar{m}=\lceil\frac{\Delta E+m_0}{\Omega}\rceil}^{\infty}\frac{\left((-\Delta E+\bar{m}\Omega)^2-m_0^2\right)^{\frac{2\bar{m}+1}{2}}r_d^{2\bar{m}}}{J_{\bar{m}}^2(\bar{m})(2\bar{m}+1)!}\times$$
$${}_1F_2\left(\frac{2\bar{m}+1}{2};\frac{2\bar{m}+3}{2},2\bar{m}+1;-((-\Delta E+\bar{m}\Omega)^2-m_0^2)r_d^2\right). \tag{2.20}$$

This gives the probability that a particle with fixed energy $\bar{\omega}=-\Delta E$ tunnels from $r_d<\frac{1}{\Omega}$ to the classical turning point $r_L=\frac{\bar{m}}{\bar{q}}>\frac{1}{\Omega}$. Note that by replacing $J_{\bar{m}}^2(\bar{m})\to 2\pi^2$ in (2.20) we

[8]This function is defined as $\lceil x\rceil:=\inf\{n\in\mathbb{Z}|\,n\geq x\}$.

obtain the total probability ρ of finding the corresponding particle at r_d:

$$\rho(-\Delta E) = \frac{1}{2\pi^2} \sum_{\bar{m}=\lceil \frac{\Delta E + m_0}{\Omega} \rceil}^{\infty} \frac{\left((-\Delta E + \bar{m}\Omega)^2 - m_0^2\right)^{\frac{2\bar{m}+1}{2}} r_d^{2\bar{m}}}{(2\bar{m}+1)!} \times$$
$$ {}_1F_2 \left(\frac{2\bar{m}+1}{2}; \frac{2\bar{m}+3}{2}, 2\bar{m}+1; -((-\Delta E + \bar{m}\Omega)^2 - m_0^2) r_d^2 \right). \quad (2.21)$$

Consequently Γ and ρ do not match as one can see in the following table[9] ($v = r_d\Omega$):

$\Delta E eV$	$\rho(-\Delta E) e \frac{eV^4}{c^4\hbar^4}$	$\Gamma(-\Delta E) \frac{eV}{c\hbar}$	$\rho(-\Delta E) e \frac{eV^4}{c^4\hbar^4}$	$\Gamma(-\Delta E) \frac{eV}{c\hbar}$
0.00	3.36×10^{-5}	3.46×10^{-3}	1.24×10^{-3}	1.65×10^{-1}
1.35	1.05×10^{-17}	2.94×10^{-15}	5.84×10^{-10}	2.17×10^{-7}
2.50	4.21×10^{-29}	1.73×10^{-36}	1.69×10^{-15}	9.03×10^{-13}
3.75	1.50×10^{-41}	8.01×10^{-39}	1.61×10^{-21}	1.11×10^{-18}
4.95	1.70×10^{-53}	1.09×10^{-50}	2.68×10^{-27}	2.20×10^{-24}
	$\Omega = \frac{\pi}{8}\frac{eV}{\hbar}$, $v = 0.1c$		$\Omega = \frac{\pi}{8}\frac{eV}{\hbar}$, $v = 0.5c$	

Table 2.1.: ρ and Γ for different values of v and Ω.

The physical reason for the mismatching of ρ and Γ comes from the finite extension of the tunneling region. Thus the normalization of the free particle wavefunction is crucial and therefore the probability of finding a particle at a certain distance from the origin is not equivalent to the tunneling probability (If we had the case where the forbidden region extends to infinity, the two probabilities would be equivalent).

Nevertheless it can be expected that the probability $\rho(-\Delta E)$ for finding a negative energy particle at the position of the detector is a good measure for the excitation of the detector. Indeed an approach using quantum field methods shows that $\rho(-\Delta E)$ is proportional to the detector's excitation rate (see Chapter 4).

[9]We have taken only the first 100 terms of the series (2.21), (2.20) into account due to their rapid convergence.

3. Quantum field theory in Minkowski spacetime

In this chapter we consider a quantized scalar field seen by an inertial and rotating observer in Minkowski spacetime. We introduce the Bogoluibov transformation and calculate the Bogoliubov coefficients between rotating and inertial frame. This implies that the vacua defined via canonical QFT in the rotating and inertial frame coincide.

3.1. Field equations in general coordinate systems

The consistent mathematical approach to relativistic quantum systems is given by field theory [Peskin and Schroeder, 2007]. In this way ambiguities do not occur in the physical interpretation of relativistic quantized systems.

The field equation for a real scalar field Φ follows from the variation principle of the action [Birrell and Davies, 1984]

$$S\left[\Phi\right] = \int \mathrm{d}^4x \frac{\sqrt{-g}}{2}\left(g^{\mu\nu}(\nabla_\mu\Phi)(\nabla_\nu\Phi) - m_0^2\Phi^2\right) =: \int \mathrm{d}^4x \mathcal{L}\left[\Phi, \partial_\mu\Phi\right], \tag{3.1}$$

with condition $\delta S = 0$. This is equivalent to the Euler-Lagrange equations for the Lagrangian density $\mathcal{L}$

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi} - \frac{\partial\mathcal{L}}{\partial\Phi} = 0. \tag{3.2}$$

For the Lagrangian above this yields

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi(x)\right) + m_0^2\Phi(x) = 0. \tag{3.3}$$

This is the same equation as for the relativistic wavefunction (B.2). But the solution needs to be interpreted as a field Φ rather than a single particle wavefunction ψ. Hence we have a system with infinitely many degrees of freedom.

3.2. Field for inertial observers

We want to solve (3.3) in an inertial frame. In Cartesian coordinates $(t, \vec{x})$, where the metric components of the Minkowski spacetime are $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, the Klein-Gordon equation (3.3) reduces to the simple form

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + m_0^2\right)\Phi(t,\vec{x}) = 0. \tag{3.4}$$

A complete[1] set of solutions of (3.4) is given in terms of plane waves:

$$f_{\vec{k}}(t,\vec{x}) = n_{\vec{k}} e^{-i\omega_{\vec{k}}t + i\vec{k}\cdot\vec{x}}, \quad \omega_{\vec{k}} := \sqrt{\vec{k}^2 + m_0^2} \tag{3.5}$$

where $n_{\vec{k}}$ is a normalization constant and will be fixed in 3.2.1.

Next we study cylindrical field modes. For this purpose it is helpful to have the solution of (3.3) in cylindrical coordinates (t, r, φ, z). It can be obtained by either inserting in (3.3) for the metric components in cylindrical coordinates $g_{\mu\nu} = \mathrm{diag}(1, -1, -r^2, -1)$ or using the fact that the KGE in the rotating frame (2.2) reduces to the correponding KGE in the inertial frame in the limit $\Omega \to 0$. By using the latter and replacing $\bar{\varphi} \to \varphi$ and $\psi \to \Phi$ the correct form of (3.3) in cylindrical coordinates reads:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} - \frac{\partial^2}{\partial z^2} + m_0^2\right)\Phi(t,r,\varphi,z) = 0. \tag{3.6}$$

Instead of solving this equation, for instance with a separation ansatz, we again take the limit $\Omega \to 0$ of the rotating wavefunction (2.8) and replace $\bar{\varphi} \to \varphi$. Moreover we denote the corresponding quantum numbers by (ω, m, k_z, q). Thus a complete set of cylindrical field modes is given by:

$$u_{\vec{q}}(t,r,\varphi,z) = N_\omega J_m(qr) e^{-i\omega t + im\varphi + ik_z z}, \quad \omega := \sqrt{q^2 + k_z^2 + m_0^2},\ \vec{q} := (q, m, k_z). \tag{3.7}$$

We see that due to the absence of the $\Omega\bar{m}$ shift the frequency ω of the mode is always positive.

3.2.1. Normalization of field modes

The field modes are normalized with respect to the scalar product (2.4), defined in Chapter 2. The normalization condition is carried over by every field mode such that the orthogonality relations for the two sets of field modes $\{f_{\vec{k}}\}$, $\{u_{\vec{q}}\}$ are

$$\left(f_{\vec{k}}, f_{\vec{k}'}\right) = \delta^3(\vec{k} - \vec{k}') \tag{3.8}$$

$$\left(u_{\vec{q}}, u_{\vec{q}'}\right) = \delta_{mm'}\delta(k_z - k_z')\frac{\delta(q - q')}{q}. \tag{3.9}$$

As in the rotating frame the spacelike hypersurface Σ_t in (2.4) can be chosen as the $\{t = 0\}$ surface. This gives $\mathrm{d}\Sigma^\mu = \mathrm{d}^3x\sqrt{-g} = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ for Cartesian coordinates so that the

[1] Completeness of $\{f_{\vec{k}}\}$ for the one-dimensional case means that any square-integrable function $F : \mathbb{R} \supseteq [a,b] \to \mathbb{C}$ can be approximated by a series $F(x) = \sum_{k=-N}^{N} c_k f_k(x)$ so that the mean square error vanishes, i.e. $\lim_{N\to\infty} \int_a^b \mathrm{d}x \left| F(x) - \sum_{k=-N}^{N} c_k f_k(x)\right|^2 = 0$. This can be generalized to arbitray dimension (cf. [Titchmarsh, 1958]).

l.h.s. of (3.8) becomes

$$in^*_{\vec{k}}n_{\vec{k}'}(-i\omega_{\vec{k}} - i\omega_{\vec{k}'}) \int \mathrm{d}x\mathrm{d}y\mathrm{d}z e^{-i\vec{x}\cdot(\vec{k}-\vec{k}')} = (2\pi)^3 n^*_{\vec{k}}n_{\vec{k}'}(\omega_{\vec{k}} + \omega_{\vec{k}'})\delta^3(\vec{k}-\vec{k}').$$

After comparing this with the r.h.s. we are able to fix the normalization constant

$$n_{\vec{k}} = \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_{\vec{k}}}}.$$

Therefore the normalized Cartesian field modes are

$$f_{\vec{k}}(t,\vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_{\vec{k}}}} e^{-i\omega_{\vec{k}}t} e^{i\vec{k}\cdot\vec{x}}. \tag{3.10}$$

The normalization constant N_ω for the cylindrical modes can be found with less effort by using the results form chapter 2. Putting $\Omega = 0$ in (2.10) and replacing $(\bar{\omega}, \bar{m}, \bar{k}_z, \bar{q}) \to (\omega, m, k_z, q)$ gives us immediately

$$u_{\vec{q}}(t,r,\varphi,z) = \frac{1}{2\pi} \frac{1}{\sqrt{2\omega}} J_m(qr) e^{-i\omega t + im\varphi + ik_z z}. \tag{3.11}$$

At this point we need to mention that besides $\{f_{\vec{k}}\}$, $\{u_{\vec{q}}\}$ their conjugates $\left\{f^*_{\vec{k}}\right\}$, $\left\{u^*_{\vec{q}}\right\}$ are a second set of linearly independent solutions with a time dependence $e^{i\omega_{\vec{k}}t}$ and $e^{i\omega t}$, respectively. The corresponding j^0 component of j^μ is negative and would not make sense as a probability density. This is the reason why j^μ is interpreted as charge density as mentioned at the end of section 2.1. In this way the continuity equation

$$\partial_\mu j^\mu = 0$$

can be considered as conservation of charge [Bjorken and Drell, 1964]. In the special case of an inertial observer the positive frequency solutions are equal to the positive charge solutions and negative frequency solutions correspond to negative charge solutions. This point of view is affirmed by the so called Feynman-Stückelberg interpretation of relativistic QM, where antiparticles propagate backwards in time and space[2] [Wachter, 2005]. Thus $f^*_{\vec{k}} \propto e^{i\omega_k t - i\vec{k}\cdot\vec{x}}$ is the antiparticle wavefunction with positive energy $\omega_{\vec{k}}$ and momentum $\vec{k}$. Analogously the same argument holds for $u^*_{\vec{q}}$ in cylindrical coordinates.

Nevertheless particles and antiparticles are indistinguishable in the absence of an external field A_μ. Therefore we only regard particles in the following[3].

[2]This interpretation is based on the CPT-invariance of the Klein-Gordon equation in an external Field A_μ.
[3]Note that for neutral charged particles (such as the photon) the corresponding antiparticle is the particle itself.

3.2.2. Field decomposition

Now we want to use the field modes $\{f_{\vec{k}}\}$, $\{u_{\vec{q}}\}$ to decompose the scalar field Φ. In general the field decomposition has the following Fourier-like form:

$$\Phi(t,\vec{x}) = \int \mathrm{d}^3k \left(a_{\vec{k}} f_{\vec{k}}(t,\vec{x}) + c_{\vec{k}} f^*_{\vec{k}}(t,\vec{x}) \right) \tag{3.12}$$

$$\Phi(t,r,\varphi,z) = \sum_{m=-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \mathrm{d}k_z \int\limits_{0}^{\infty} \mathrm{d}qq \left(b_{\vec{q}} u_{\vec{q}}(t,r,\varphi,z) + d_{\vec{q}} u^*_{\vec{q}}(t,r,\varphi,z) \right). \tag{3.13}$$

Since we consider a real scalar field (i.e. neutral particles), the coefficients are related by

$$c_{\vec{k}} = a^*_{\vec{k}} \quad \text{and} \quad d_{\vec{q}} = b^*_{\vec{q}},$$

so that the field decomposition reads

$$\Phi(t,\vec{x}) = \int \mathrm{d}^3k \left(a_{\vec{k}} f_{\vec{k}}(t,\vec{x}) + a^*_{\vec{k}} f^*_{\vec{k}}(t,\vec{x}) \right) \tag{3.14}$$

$$\Phi(t,r,\varphi,z) = \sum_{m=-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \mathrm{d}k_z \int\limits_{0}^{\infty} \mathrm{d}qq \left(b_{\vec{q}} u_{\vec{q}}(t,r,\varphi,z) + b^*_{\vec{q}} u^*_{\vec{q}}(t,r,\varphi,z) \right). \tag{3.15}$$

For the quantization later on we need the conjugate momentum Π, which is defined by [Birrell and Davies, 1984]

$$\Pi := \frac{\partial \mathcal{L}}{\partial \partial_0 \Phi} = \sqrt{-g} g^{0\mu} \partial_\mu \Phi. \tag{3.16}$$

Inserting (3.14) and (3.15) in (3.16) gives:

$$\Pi(t,\vec{x}) = i \int \mathrm{d}^3k \omega_{\vec{k}} \left(a_{\vec{k}} f_{\vec{k}}(t,\vec{x}) - a^*_{\vec{k}} f^*_{\vec{k}}(t,\vec{x}) \right) \tag{3.17}$$

$$\Pi(t,r,\varphi,z) = -ir \sum_{m=-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \mathrm{d}k_z \int\limits_{0}^{\infty} \mathrm{d}qq\omega \left(b_{\vec{q}} u_{\vec{q}}(t,r,\varphi,z) - b^*_{\vec{q}} u^*_{\vec{q}}(t,r,\varphi,z) \right). \tag{3.18}$$

3.3. Field decomposition for rotating observers

For the field decomposition in the rotating frame $(t,r,\bar{\varphi},z)$ we use the complete set $\{\psi_{\vec{\bar{q}}}\}$ to expand the scalar field Φ. This set consists of all free particle wavefunctions of the rotating frame (2.11). Thus we have[4]

$$\Phi(t,r,\bar{\varphi},z) = \sum_{\bar{m}=-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \mathrm{d}\bar{k}_z \int\limits_{0}^{\infty} \mathrm{d}\bar{q}\bar{q} \left(c_{\vec{\bar{q}}} \psi_{\vec{\bar{q}}}(t,r,\bar{\varphi},z) + c^*_{\vec{\bar{q}}} \psi^*_{\vec{\bar{q}}}(t,r,\bar{\varphi},z) \right). \tag{3.19}$$

[4]The transformation from cylindrical into rotating coordinates changes only the energy of the field modes. The quantum numbers of the inertial frame q, m, k_z have the same values in the rotating frame. This means that $q = \bar{q}, m = \bar{m}, k_z = \bar{k}_z$. Nevertheless we will use the symbols with a bar to distinguish explicitly between rotating and non rotating frame.

The field momentum (3.16) has the form

$$\Pi = r(\partial_0\Phi - \Omega\partial_{\bar{\varphi}}\Phi) \tag{3.20}$$

so that it can be expressed as

$$\Pi(t,r,\bar{\varphi},z) = -ir\sum_{\bar{m}=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\bar{k}_z\int_{0}^{\infty}\mathrm{d}\bar{q}\bar{q}(\bar{\omega}+\bar{m}\Omega)\left(c_{\vec{\bar{q}}}\psi_{\vec{\bar{q}}}(t,r,\bar{\varphi},z) - c^*_{\vec{\bar{q}}}\psi^*_{\vec{\bar{q}}}(t,r,\bar{\varphi},z)\right). \tag{3.21}$$

The similarity with the cylindrical mode expansions (3.15) and (3.18) is obvious. In fact up to normalization the field decomposition can be obtained from the cylindrical one by

$$(\omega,q,m,k_z) \to (\bar{\omega},\bar{q},\bar{m},\bar{k}_z) \quad \text{and} \quad \varphi \to \bar{\varphi}. \tag{3.22}$$

Nevertheless there is a crucial difference: $\{\psi_{\vec{\bar{q}}}\}$ may carry negative frequency although the corresponding charge is always positive. The converse holds for $\psi^*_{\vec{\bar{q}}}$. The consequence is that the Hamiltonian H of the field in the rotating frame is not bounded from below. This statement is proven in the following section.

3.4. Hamiltonian of the field in the rotating frame

The Hamiltonian H of the field Φ is given by [Letaw and Pfautsch, 1981]

$$H = \int_{\Sigma_t}\mathrm{d}\Sigma g^{0\nu}X^\mu T_{\mu\nu}, \quad X^\mu = (1,0,0,0),\ \mathrm{d}\Sigma := \mathrm{d}^3x\sqrt{-g}, \tag{3.23}$$

where $T_{\mu\nu}$ denotes the stress-energy-momentum tensor of the field Φ. In order to show that H is not bounded from below in the rotating frame we insert the general form of $T_{\mu\nu}$ [Birrell and Davies, 1984]

$$T_{\mu\nu} = \Phi_{,\mu}\Phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\left(\Phi_{,\sigma}\Phi^{,\sigma} - m_0^2\Phi^2\right) \tag{3.24}$$

and the metric of the rotating frame (1.7) into the integrand of (3.23). This yields

$$\begin{aligned} g^{0\nu}T_{0\nu} &= T_{00}g^{00} + T_{02}g^{02} = \Phi_{,0}\, g^{00} + \Phi_{,0}\Phi_{,2}\, g^{02} - \frac{1}{2}\left(g_{00}g^{00} + g_{02}g^{02}\right)\left(\Phi_{,\sigma}\Phi^{,\sigma} - m_0^2\Phi^2\right) \\ &= \Phi_{,0}^2 - \Omega\Phi_{,0}\Phi_{,2} - \frac{1}{2}\left(\Phi_{,0}^2 - 2\Omega\Phi_{,0}\Phi_{,2} - \Phi_{,1}^2 - \frac{1}{r^2}\Phi_{,2}^2 - \Omega^2\Phi_{,2}^2 - \Phi_{,3}^2 - m_0^2\Phi^2\right) \\ &= \frac{1}{2}\left(\Phi_{,0}^2 + \Phi_{,1}^2 + \frac{1}{r^2}\Phi_{,2}^2 - \Omega^2\Phi_{,2}^2 + \Phi_{,3}^2 + m_0^2\Phi^2\right). \end{aligned}$$

In the following we use the abbreviation $\sum_{\vec{\bar{q}}} := \sum_{\bar{m}=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\bar{k}_z\int_0^\infty \mathrm{d}\bar{q}\bar{q}$ so that the field decomposition (3.19) reads $\Phi = \sum_{\vec{\bar{q}}}(c_{\vec{\bar{q}}}\psi_{\vec{\bar{q}}} + c^*_{\vec{\bar{q}}}\psi^*_{\vec{\bar{q}}})$. Consequently the first term of the integrand gives

$$\Phi,_0^2 = \sum_{\vec{q}} \sum_{\vec{q}'} (-\bar{\omega}\bar{\omega}') \left(c_{\vec{q}} c_{\vec{q}'} \psi_{\vec{q}} \psi_{\vec{q}'} + c^*_{\vec{q}} c^*_{\vec{q}'} \psi^*_{\vec{q}} \psi^*_{\vec{q}'} - c^*_{\vec{q}} c_{\vec{q}'} \psi^*_{\vec{q}} \psi_{\vec{q}'} - c_{\vec{q}} c^*_{\vec{q}'} \psi_{\vec{q}} \psi^*_{\vec{q}'} \right).$$

Note that

$$\begin{aligned}
\int\limits_{\Sigma_t} \mathrm{d}\Sigma \psi_{\vec{q}} \psi_{\vec{q}'} &= \frac{1}{(2\pi)^2} \frac{1}{2\sqrt{(\bar{\omega} + \bar{m}\Omega)(\bar{\omega}' + \bar{m}'\Omega)}} \underbrace{\int\limits_{-\infty}^{\infty} \mathrm{d}z e^{iz(\bar{k}_z + \bar{k}'_z)}}_{2\pi\delta(\bar{k}_z + \bar{k}'_z)} \underbrace{\int\limits_0^{2\pi} e^{i\bar{\varphi}(\bar{m}+\bar{m}')}}_{2\pi\delta_{\bar{m}(-\bar{m}')}} \underbrace{\int\limits_0^{\infty} \mathrm{d}r r J_{\bar{m}}(\bar{q}r) J_{\bar{m}'}(\bar{q}'r)}_{(-1)^{\bar{m}} \bar{q}^{-1} \delta(\bar{q} - \bar{q}')} \\
&= (-1)^{\bar{m}} \left. \frac{1}{2\omega} \right|_{\vec{q} = (\bar{q}, -\bar{m}, -\bar{k}_z)} \delta_{\bar{m}(-\bar{m}')} \delta(\bar{k}_z + \bar{k}'_z) \frac{\delta(\bar{q} - \bar{q}')}{\bar{q}} \\
&= \int\limits_{\Sigma_t} \mathrm{d}\Sigma \psi^*_{\vec{q}} \psi^*_{\vec{q}'} \\
\int\limits_{\Sigma_t} \mathrm{d}\Sigma \psi^*_{\vec{q}} \psi_{\vec{q}'} &= \left. \frac{1}{2\omega} \right|_{\vec{q} = (\bar{q}, \bar{m}, \bar{k}_z)} \delta_{\bar{m}\bar{m}'} \delta(\bar{k}_z - \bar{k}'_z) \frac{\delta(\bar{q} - \bar{q}')}{\bar{q}} = \int\limits_{\Sigma_t} \mathrm{d}\Sigma \psi_{\vec{q}} \psi^*_{\vec{q}'}.
\end{aligned}$$

Therefore the first term of the Hamiltonian gives

$$\begin{aligned}
\frac{1}{2} \int\limits_{\Sigma_t} \mathrm{d}\Sigma \Phi,_0^2 &= \frac{1}{2} \sum_{\vec{q}\vec{q}'} \frac{-\bar{\omega}\bar{\omega}'}{2\sqrt{m_0^2 + \bar{k}_z^2 + \bar{q}^2}} \Bigg((c_{\vec{q}} c_{\vec{q}'} + c^*_{\vec{q}} c^*_{\vec{q}'})(-1)^{\bar{m}} \delta_{\bar{m}(-\bar{m}')} \delta(\bar{k}_z + \bar{k}'_z) \frac{\delta(\bar{q} - \bar{q}')}{\bar{q}} \\
&\quad - (c^*_{\vec{q}} c_{\vec{q}'} + c_{\vec{q}} c^*_{\vec{q}'}) \delta_{\bar{m}\bar{m}'} \delta(\bar{k}_z - \bar{k}'_z) \frac{\delta(\bar{q} - \bar{q}')}{\bar{q}} \Bigg) \\
&= \frac{1}{4} \sum_{\vec{q}} \Bigg(- \left(\sqrt{m_0^2 + \bar{q}^2 + \bar{k}_z^2} - \frac{\bar{m}^2 \Omega^2}{\sqrt{m_0^2 + \bar{q}^2 + \bar{k}_z^2}} \right) (-1)^{\bar{m}} \left. \left(c_{\vec{q}} c_{\vec{q}'} + c^*_{\vec{q}} c^*_{\vec{q}'} \right) \right|_{\vec{q}' = (\bar{q}, -\bar{m}, -\bar{k}_z)} \\
&\quad + \left(\sqrt{m_0^2 + \bar{q}^2 + \bar{k}_z^2} + \frac{\bar{m}^2 \Omega^2}{\sqrt{m_0^2 + \bar{q}^2 + \bar{k}_z^2}} - 2\bar{m}\Omega \right) \left(c^*_{\vec{q}} c_{\vec{q}} + c_{\vec{q}} c^*_{\vec{q}} \right) \Bigg).
\end{aligned}$$

In the next term $\frac{1}{2} \int\limits_{\Sigma_t} \mathrm{d}\Sigma \Phi,_1^2$ terms of the following type appear:

$$\int\limits_{\Sigma_t} \mathrm{d}\Sigma \psi_{\vec{q}},_1 \psi_{\vec{q}'},_1 = \frac{\delta_{\bar{m}(-\bar{m}')} \delta(\bar{k}_z + \bar{k}'_z)}{2\sqrt{(\bar{\omega} + \bar{m}\Omega)(\bar{\omega}' + \bar{m}'\Omega)}} \bar{q}\bar{q}' \int\limits_0^{\infty} \mathrm{d}r J'_{\bar{m}}(\bar{q}r) J'_{\bar{m}'}(\bar{q}'r).$$

In order to evaluate the integral on the r.h.s. we use [Korenev, 2002] (p.14)

$$\begin{aligned}
J'_{\bar{m}}(z) &= \frac{1}{2} \left(J_{\bar{m}-1}(z) - J_{\bar{m}+1}(z) \right) \\
\frac{2\bar{m}}{z} J_{\bar{m}}(z) &= J_{\bar{m}-1}(z) + J_{\bar{m}+1}(z), \quad \forall \bar{m}, z \in \mathbb{C}
\end{aligned}$$

so that

$$\bar{q}\bar{q}'\int_0^\infty \mathrm{d}r J'_{\bar{m}}(\bar{q}r)J'_{-\bar{m}}(\bar{q}'r) = \frac{(-1)^{\bar{m}}}{4}\bar{q}\bar{q}'\underbrace{\int_0^\infty \mathrm{d}rr\left(J_{\bar{m}-1}(\bar{q}r)J_{\bar{m}-1}(\bar{q}'r) + J_{\bar{m}+1}(\bar{q}r)J_{\bar{m}+1}(\bar{q}'r)\right)}_{2\bar{q}'^{-1}\delta(\bar{q}-\bar{q}')}$$
$$-\frac{(-1)^{\bar{m}}}{4}\bar{q}\bar{q}'\underbrace{\int_0^\infty \mathrm{d}r\left(J_{\bar{m}-1}(\bar{q}r)J_{\bar{m}+1}(\bar{q}'r) + J_{\bar{m}+1}(\bar{q}r)J_{\bar{m}-1}(\bar{q}'r)\right)}_{-2\bar{q}'^{-1}\delta(\bar{q}-\bar{q}')+\int_0^\infty \mathrm{d}rr\frac{4\bar{m}^2}{\bar{q}\bar{q}'r^2}J_{\bar{m}}(\bar{q}r)J_{\bar{m}}(\bar{q}'r)}$$
$$= (-1)^{\bar{m}}\bar{q}\delta(\bar{q}-\bar{q}') - (-1)^{\bar{m}}\int_0^\infty \mathrm{d}rr\frac{\bar{m}^2}{r^2}J_{\bar{m}}(\bar{q}r)J_{\bar{m}}(\bar{q}'r).$$

The last term appears as a factor in the third term $\int \mathrm{d}\Sigma\Phi,_2 r^{-2}$ with a positive sign. Thus we have for the second and third term of H:

$$\frac{1}{2}\int_{\Sigma_t}\mathrm{d}\Sigma\left(\Phi,_1^2 + \frac{1}{r^2}\Phi,_2^2\right) = \frac{1}{2}\sum_{\vec{\bar{q}}\vec{\bar{q}}'}\left(\frac{\bar{q}\delta_{\bar{m}(-\bar{m}')}\delta(\bar{k}_z+\bar{k}'_z)\delta(\bar{q}-\bar{q})}{2\sqrt{m_0^2+\bar{k}_z^2+\bar{q}^2}}(-1)^{\bar{m}}(c_{\vec{\bar{q}}}c_{\vec{\bar{q}}'} + c^*_{\vec{\bar{q}}}c^*_{\vec{\bar{q}}'})|_{\vec{\bar{q}}'=(\bar{q},-\bar{m},-\bar{k}_z)}\right.$$
$$\left. + \frac{\bar{q}\delta_{\bar{m}\bar{m}'}\delta(\bar{k}_z-\bar{k}'_z)\delta(\bar{q}-\bar{q})}{2\sqrt{m_0^2+\bar{k}_z^2+\bar{q}^2}}(c^*_{\vec{\bar{q}}}c_{\vec{\bar{q}}} + c_{\vec{\bar{q}}}c^*_{\vec{\bar{q}}})\right)$$
$$= \frac{1}{4}\sum_{\vec{\bar{q}}}\frac{\bar{q}^2}{\sqrt{m_0^2+\bar{k}_z^2+\bar{q}^2}}\left((-1)^{\bar{m}}(c_{\vec{\bar{q}}}c_{\vec{\bar{q}}'} + c^*_{\vec{\bar{q}}}c^*_{\vec{\bar{q}}'})|_{\vec{\bar{q}}'=(\bar{q},-\bar{m},-\bar{k}_z)} + c^*_{\vec{\bar{q}}}c_{\vec{\bar{q}}} + c_{\vec{\bar{q}}}c^*_{\vec{\bar{q}}}\right).$$

Analogously the remaining terms yield

$$\frac{1}{2}\int_{\Sigma_t}\mathrm{d}\Sigma(-\Omega^2\Phi,_2^2 + \Phi,_3^2 + m_0^2\Phi^2) = \frac{1}{4}\sum_{\vec{\bar{q}}}\frac{-\bar{m}^2\Omega^2+\bar{k}_z^2+m_0^2}{\sqrt{m_0^2+\bar{k}_z^2+\bar{q}^2}}$$
$$\times\left((-1)^{\bar{m}}(c_{\vec{\bar{q}}}c_{\vec{\bar{q}}'} + c^*_{\vec{\bar{q}}}c^*_{\vec{\bar{q}}'})|_{\vec{\bar{q}}'=(\bar{q},-\bar{m},-\bar{k}_z)} + c^*_{\vec{\bar{q}}}c_{\vec{\bar{q}}} + c_{\vec{\bar{q}}}c^*_{\vec{\bar{q}}}\right).$$

Collecting the results above we obtain for H:

$$H = \frac{1}{4}\sum_{\vec{\bar{q}}}\left(\left(-\sqrt{m_0^2+\bar{k}_z^2+\bar{q}^2} + \frac{\bar{m}^2\Omega}{\sqrt{m_0^2+\bar{k}_z^2+\bar{q}^2}} + \frac{\bar{q}^2-\bar{m}^2\Omega^2+\bar{k}_z^2+m_0^2}{\sqrt{m_0^2+\bar{k}_z^2+\bar{q}^2}}\right)\right.$$
$$\times(-1)^{\bar{m}}(c_{\vec{\bar{q}}}c_{\vec{\bar{q}}'} + c^*_{\vec{\bar{q}}}c^*_{\vec{\bar{q}}'})|_{\vec{\bar{q}}'=(\bar{q},-\bar{m},-\bar{k}_z)}$$
$$\left. + \left(\sqrt{m_0^2+\bar{k}_z^2+\bar{q}^2} + \frac{\bar{m}^2\Omega^2}{\sqrt{m_0^2+\bar{k}_z^2+\bar{q}^2}} - 2\bar{m}\Omega + \frac{\bar{q}^2-\bar{m}^2\Omega^2+\bar{k}_z^2+m_0^2}{\sqrt{m_0^2+\bar{k}_z^2+\bar{q}^2}}\right)(c^*_{\vec{\bar{q}}}c_{\vec{\bar{q}}} + c_{\vec{\bar{q}}}c^*_{\vec{\bar{q}}})\right)$$

$$\Longrightarrow H = \frac{1}{2}\sum_{\bar{m}=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\bar{k}_z\int_0^\infty \mathrm{d}\bar{q}\bar{q}\left(\sqrt{m_0^2+\bar{k}_z^2+\bar{q}^2} - \bar{m}\Omega\right)(c^*_{\vec{\bar{q}}}c_{\vec{\bar{q}}} + c_{\vec{\bar{q}}}c^*_{\vec{\bar{q}}}) \tag{3.25}$$

This shows that the Hamiltonian is not bounded from below in the rotating system.

3.5. Canonical quantization procedure

There are many methods available for quantizing a field. For instance the quantization procedure of Feynman and Dirac (which makes use of the path integral) provides a physically intuitive approach to the subject. The simplest and conceptually clearest quantization method is the so called canonical quantization. This method (developed by Heisenberg, Born and Jordan) is close to the quantization of Hamiltonian mechanics of systems with a finite number of degrees of freedom and is essential for practical calculations [Scheck, 2007]. This is the reason why we choose this procedure in the following.

The quantization of a field (i.e. system with infinitely many degrees of freedom) starts with the promotion of the field obervables Φ, Π to operators $\hat{\Phi}, \hat{\Pi}$. These operators satisfy the canonical commutation relations[5] ($\hbar = 1$):

$$[\hat{\Phi}(x), \hat{\Pi}(x')]|_{t=t'} = \begin{cases} i\delta^{(3)}(\vec{x}-\vec{x}') & \text{for Cartesian coordinates} \\ i\delta(r-r')\delta(\varphi-\varphi')\delta(z-z') & \text{for cylindrical coordinates} \\ i\delta(r-r')\delta(\bar{\varphi}-\bar{\varphi}')\delta(z-z') & \text{for rotating coordinates} \end{cases} \tag{3.26}$$

$$[\hat{\Phi}(x), \hat{\Phi}(x')]|_{t=t'} = 0 = [\hat{\Pi}(x), \hat{\Pi}(x')]|_{t=t'} \tag{3.27}$$

By inserting the appropriate mode expansions the coefficients $a_{\vec{k}}, b_{\vec{q}}, c_{\vec{\bar{q}}}$ become operators themselves. The imposed commutation relations are equivalent to the following relations for the operators $\hat{a}_{\vec{k}}, \hat{b}_{\vec{q}}, \hat{c}_{\vec{\bar{q}}}$ and their Hermitian conjugates $\hat{a}^\dagger_{\vec{k}}, \hat{b}^\dagger_{\vec{q}}, \hat{c}^\dagger_{\vec{\bar{q}}}$:

$$[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] = \left(f_{\vec{k}}, f_{\vec{k}'}\right) = \delta^{(3)}(\vec{k}-\vec{k}'), \quad [\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}] = 0 = [\hat{a}^\dagger_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] \tag{3.28}$$

$$[\hat{b}_{\vec{q}}, \hat{b}^\dagger_{\vec{q}'}] = (u_{\vec{q}}, u_{\vec{q}'}) = \delta_{mm'}\delta(k_z-k'_z)\frac{\delta(q-q')}{q}, \quad [\hat{b}_{\vec{q}}, \hat{b}_{\vec{q}'}] = 0 = [\hat{b}^\dagger_{\vec{q}}, \hat{b}^\dagger_{\vec{q}'}] \tag{3.29}$$

$$[\hat{c}_{\vec{\bar{q}}}, \hat{c}^\dagger_{\vec{\bar{q}}'}] = \left(\psi_{\vec{\bar{q}}}, \psi_{\vec{\bar{q}}'}\right) = \delta_{\bar{m}\bar{m}'}\delta(\bar{k}_z-\bar{k}'_z)\frac{\delta(\bar{q}-\bar{q}')}{\bar{q}}, \quad [\hat{c}_{\vec{\bar{q}}}, \hat{c}_{\vec{\bar{q}}'}] = 0 = [\hat{c}^\dagger_{\vec{\bar{q}}}, \hat{c}^\dagger_{\vec{\bar{q}}'}] \tag{3.30}$$

We prove this for the rotating system (3.30) only, since for Cartesian modes and cylindrical modes this can be found in [Scheck, 2007] for instance. First we start by showing that

$$\hat{c}_{\vec{\bar{q}}} = \left(\psi_{\vec{\bar{q}}}, \hat{\Phi}\right) \quad \hat{c}^\dagger_{\vec{\bar{q}}} = -\left(\psi^*_{\vec{\bar{q}}}, \hat{\Phi}\right)$$

$$\begin{aligned} \left(\psi_{\vec{\bar{q}}}, \hat{\Phi}\right) &= i\int \mathrm{d}\Sigma^\mu \psi^*_{\vec{\bar{q}}} \overleftrightarrow{\partial_\mu} \hat{\Phi} \\ &= \sum_{\bar{m}'} \int \mathrm{d}\bar{k}_z \int \mathrm{d}\bar{q}\bar{q} \left(\hat{c}_{\vec{\bar{q}}'} \underbrace{\left(\psi_{\vec{\bar{q}}}, \psi_{\vec{\bar{q}}'}\right)}_{\delta_{\bar{m}\bar{m}'}\delta(\bar{k}_z-\bar{k}'_z)\frac{\delta(\bar{q}-\bar{q}')}{\bar{q}}} + \hat{c}^\dagger_{\vec{\bar{q}}'} \underbrace{\left(\psi_{\vec{\bar{q}}}, \psi^*_{\vec{\bar{q}}'}\right)}_{0} \right) \\ &= \hat{c}_{\vec{\bar{q}}} \end{aligned}$$

[5] A classical Hamiltonian system (q_i, p_i) has the Poission brackets $\{q_i, p_j\} = \delta_{ij}$. Quantization starts with the replacements: $(q_i, p_j) \to (\hat{q}_i, \hat{p}_j)$ and $\{,\} \to -\frac{i}{\hbar}[,]$.

The other relation follows from the hermiticity $\left(\psi_{\vec{q}}, \hat{\Phi}\right)^\dagger = -\left(\psi^*_{\vec{q}}, \hat{\Phi}\right)$ of the inner product. In the following we suppress the arguments of the mode functions and field operators by writing $\psi'_{\vec{q}}$ instead of $\psi_{\vec{q}}(t, r', \bar{\varphi}', z')$ (analogously for $\hat{\Phi}$). Then we have:

$$
\begin{aligned}
[\hat{c}_{\vec{q}}, \hat{c}^\dagger_{\vec{q}'}] &= -\left[\left(\psi_{\vec{q}}, \hat{\Phi}\right), \left(\psi^*_{\vec{q}'}, \hat{\Phi}\right)\right] \\
&= -i^2 \int \mathrm{d}\Sigma^\mu \int \mathrm{d}\Sigma'^\nu \left[\psi^*_{\vec{q}} \overleftrightarrow{\partial_\mu} \hat{\Phi}, \psi'_{\vec{q}'} \overleftrightarrow{\partial'_\nu} \hat{\Phi}'\right] \\
&= \int \mathrm{d}\Sigma^\mu \int \mathrm{d}\Sigma'^\nu \psi^*_{\vec{q}} \psi'_{\vec{q}'} \underbrace{\overleftrightarrow{\partial_\mu} \overleftrightarrow{\partial'_\nu}}_{(\overrightarrow{\partial_\mu} - \overleftarrow{\partial_\mu})(\overrightarrow{\partial'_\nu} - \overleftarrow{\partial'_\nu})} \left[\hat{\Phi}, \hat{\Phi}'\right] \\
&= \int \mathrm{d}\Sigma^\mu \int \mathrm{d}\Sigma'^\nu \psi^*_{\vec{q}} \psi'_{\vec{q}'} \left(\overrightarrow{\partial_\mu}\overrightarrow{\partial'_\nu} - \overleftarrow{\partial_\mu}\overrightarrow{\partial'_\nu} - \overrightarrow{\partial_\mu}\overleftarrow{\partial'_\nu} + \overleftarrow{\partial_\mu}\overleftarrow{\partial'_\nu}\right) \left[\hat{\Phi}, \hat{\Phi}'\right] \\
&= \int \mathrm{d}\Sigma^\mu \int \mathrm{d}\Sigma'^\nu \psi^*_{\vec{q}} \psi'_{\vec{q}'} \left[\partial_\mu \hat{\Phi}, \partial'_\nu \hat{\Phi}'\right] \\
&- \int \mathrm{d}\Sigma^\mu \int \mathrm{d}\Sigma'^\nu (\partial_\mu \psi^*_{\vec{q}}) \psi'_{\vec{q}'} \left[\hat{\Phi}, \partial'_\nu \hat{\Phi}'\right] \\
&- \int \mathrm{d}\Sigma^\mu \int \mathrm{d}\Sigma'^\nu \psi^*_{\vec{q}} (\partial'_\nu \psi'_{\vec{q}'}) \left[\partial_\mu \hat{\Phi}, \hat{\Phi}'\right] \\
&+ \int \mathrm{d}\Sigma^\mu \int \mathrm{d}\Sigma'^\nu (\partial_\mu \psi^*_{\vec{q}})(\partial'_\nu \psi'_{\vec{q}'}) \left[\hat{\Phi}, \hat{\Phi}'\right]
\end{aligned}
$$

Using the general definition for the field momentum (3.16) and the commutation relations (3.26) we see that the nonvanishing contribution (second and third term) is:

$$
\begin{aligned}
&\int \mathrm{d}\Sigma^\mu (-\partial_\mu \psi^*_{\vec{q}}) \int \mathrm{d}z' d\bar{\varphi}' \mathrm{d}r' \psi'_{\vec{q}'} i\delta(r - r')\delta(\bar{\varphi} - \bar{\varphi}')\delta(z - z') \\
&+ \int \mathrm{d}\Sigma'^\nu (\partial'_\nu \psi'_{\vec{q}'}) \int \mathrm{d}z d\bar{\varphi} \mathrm{d}r (-\psi^*_{\vec{q}})(-i)\delta(r - r')\delta(\bar{\varphi} - \bar{\varphi}')\delta(z - z') \\
&= -i \underbrace{\left(\int \mathrm{d}\Sigma^\mu \; \psi'_{\vec{q}'} (\partial_\mu \psi^*_{\vec{q}}) - \int \mathrm{d}\Sigma'^\nu (\partial'_\nu \psi'_{\vec{q}'}) \psi^*_{\vec{q}}\right)}_{-\frac{1}{i}\left(\psi_{\vec{q}'}, \psi_{\vec{q}}\right)} \\
&= \delta_{\bar{m}'\bar{m}} \delta(\bar{k}'_z - \bar{k}_z) \frac{\delta(\bar{q}' - \bar{q})}{\bar{q}'}
\end{aligned}
$$

With the help of $\left(\psi^*_{\vec{q}'}, \psi_{\vec{q}}\right) = 0$ one obtains analogously the other two vanishing commutation relations of (3.30).

So far we have not mentioned how to interpret the operators $\hat{c}_{\vec{q}}$, $\hat{c}^\dagger_{\vec{q}}$ physically. This is done in the following section.

3.6. Vacuum state via canonical quantum field theory

The canonical quantization procedure for fields is done in the Heisenberg picture. In this picture the time dependence of the considered quantum system is entirely contained in the operators. The corresponding quantum states $|\psi\rangle$ span a Hilbert space and are time independent [Scheck, 2007].

In inertial systems a convenient basis for the Hilbert space is the so called Fock representation. In this representation the vacuum state $|0\rangle$ is defined by:

$$\hat{a}_{\vec{k}} |0\rangle = 0, \quad \forall \vec{k} \tag{3.31}$$

and is normalized according to

$$\langle 0| \, 0\rangle = 1.$$

This state is fundamental since every other state, even multiparticle states, can be created out of $|0\rangle$. For instance the state $|1_{\vec{k}}\rangle$ can be constructed by letting $\hat{a}^\dagger_{\vec{k}}$ act on the vacuum state

$$|1_{\vec{k}}\rangle = \hat{a}^\dagger_{\vec{k}} |0\rangle . \tag{3.32}$$

Physically $|1_{\vec{k}}\rangle$ represents a particle with momentum $\vec{k}$. Arbitrary many-particle states may be constructed similarly by

$$\left| n_{\vec{k}_1}, n_{\vec{k}_2}, \cdots \right\rangle = \prod_{j=1}^{\infty} \left(n_{\vec{k}_j}! \right)^{-\frac{1}{2}} \left(\hat{a}^\dagger_{\vec{k}_j} \right)^{n_{\vec{k}_j}} |0\rangle . \tag{3.33}$$

(3.33) represents a state with $n_{\vec{k}_1}$ particles with momentum $\vec{k}_1$, $n_{\vec{k}_2}$ particles with momentum $\vec{k}_2$, and so on.

States of this type are the eigenstates of the number operator $\hat{n}_{\vec{k}}$ (counting the number of quanta in mode $\vec{k}$) which is defined by

$$\hat{n}_{\vec{k}} := \hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} \tag{3.34}$$

with

$$\hat{n}_{\vec{k}} \left| n_{\vec{k}'} \right\rangle = \delta_{\vec{k}\vec{k}'} n_{\vec{k}} \left| n_{\vec{k}'} \right\rangle , \quad n_{\vec{k}} \in \mathbb{N}.$$

Note that due to the commutation relation $[\hat{a}^\dagger_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] = 0$ the multi-particle state (3.33) obeys Bose statistics. This means that the state (3.33) is symmetric under exchange of two arbitrary particles. Therefore scalar fields with commutation relations of the form (3.26) represent bosons[6].

The definition of the one particle state (3.32) looks similar to the first excited state of the one-dimensional quantum harmonic oscillator. Also the commutation relations, satisfied by $\hat{a}^\dagger_{\vec{k}}$ and $\hat{a}_{\vec{k}}$, are similar to those satisfied by the creation and annihilation operators of the harmonic oscillator[7]. Hence it seems natural to call $\hat{a}^\dagger_{\vec{k}}$ the creation operator for a particle with momentum $\vec{k}$ and $\hat{a}_{\vec{k}}$ the corresponding annihilation operator. Indeed one can show that the quantum scalar field can be considered as a collection of infinitely many quantum harmonical oscillators [Peskin and Schroeder, 2007]. This mirrors that QFT is a

[6] The use of anticommutation relations (i.e. replacing $[,] \to \{,\}$) would describe fermions [Parker and Toms, 2009].

[7] The difference is that in contrast to the n-dimensional harmonic oscillator the index $\vec{k}$, labeling the operators $\hat{a}^\dagger_{\vec{k}}$, $\hat{a}_{\vec{k}}$, is continuous. Thus instead of having the discrete Kronecker delta one has the Dirac delta on the r. h. s. of the commutation relations.

many particle theory.

By considering (3.32) and (3.33) one notes that those particles are just excitations of $|0\rangle$. Therefore the particle concept in QFT relies completely on the choice of the generalized basis $\{f_{\vec{k}}\}$ of the field decomposition.

If we choose $\{u_{\vec{q}}\}$ to define the 'cylindrical' vacuum state $|\tilde{0}\rangle$ via

$$\hat{b}_{\vec{q}}\,|\tilde{0}\rangle = 0, \quad \forall \vec{q},$$

it is a priori not clear that we are allowed to identify 'cylindrical' particles (obtained by letting $\hat{b}^{\dagger}_{\vec{q}}$ operate on $|\tilde{0}\rangle$) with Cartesian ones (3.33). Of course from the physical point of view they should be equivalent, since the bases represent solutions of (3.3), which separate in coordinates that differ only by a spatial coordinate transformation. Indeed the equivalence of $|\tilde{0}\rangle$ and $|0\rangle$ is shown in 3.7.1.

On the other hand if $|0\rangle_R$ denotes the vacuum state corresponding to the set $\{\psi_{\vec{q}}\}$, it cannot be expected that $|0\rangle_R = |0\rangle$. This implies that a priori the rotating observer has a different particle concept from the inertial one. How $|0\rangle_R$ is related to the Minkowski vacuum $|0\rangle$ was discussed at the beginning of the 1980s by Letaw and Pfautsch (see [Letaw and Pfautsch, 1980]). They found a surprising result, which we reproduce explicitly in 3.7.2.

3.7. Bogoliubov transformation

Two generalized bases of the field, e.g. $\{f_{\vec{k}}\}$ and $\{u_{\vec{q}}\}$, are related by [Birrell and Davies, 1984]:

$$f_{\vec{k}} = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}k_z \int_0^{\infty} \mathrm{d}q q \left(\alpha_{\vec{k}\vec{q}} u_{\vec{q}} + \beta_{\vec{k}\vec{q}} u^*_{\vec{q}}\right) \tag{3.35}$$

$$u_{\vec{q}} = \int \mathrm{d}^3k \left(\alpha^*_{\vec{k}\vec{q}} f_{\vec{k}} - \beta_{\vec{k}\vec{q}} f^*_{\vec{k}}\right). \tag{3.36}$$

(3.35) and (3.36) are called Bogoliubov transformations and the coefficients $\alpha_{\vec{k}\vec{q}}$, $\beta_{\vec{k}\vec{q}}$ are called Bogoliubov coefficients. We show that they can be obtained from

$$\alpha_{\vec{k}\vec{q}} = \left(u_{\vec{q}}, f_{\vec{k}}\right) \quad \text{and} \quad \beta_{\vec{k}\vec{q}} = -\left(u^*_{\vec{q}}, f_{\vec{k}}\right). \tag{3.37}$$

Proof:

$$\begin{aligned}
\left(u_{\vec{q}}, f_{\vec{k}}\right) &= \sum_{m'=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}k'_z \int_0^{\infty} \mathrm{d}q' q' \left(u_{\vec{q}}, \alpha_{\vec{k}\vec{q}'} u_{\vec{q}'} + \beta_{\vec{k}\vec{q}'} u^*_{\vec{q}'}\right) \\
&= \sum_{m'=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}k'_z \int_0^{\infty} \mathrm{d}q' q' \alpha_{\vec{k}\vec{q}'} \underbrace{\left(u_{\vec{q}}, u_{\vec{q}'}\right)}_{\delta_{mm'}\delta(k_z - k'_z)\frac{\delta(q-q')}{q'}} + \sum_{m'=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}k'_z \int_0^{\infty} \mathrm{d}q' q' \beta_{\vec{k}\vec{q}'} \underbrace{\left(u_{\vec{q}}, u^*_{\vec{q}'}\right)}_{0} \\
&= \alpha_{\vec{k}\vec{q}}
\end{aligned}$$

For $\beta_{\vec{k}\vec{q}}$ we have

$$\begin{aligned}
-\left(u^*_{\vec{q}}, f_{\vec{k}}\right) &= -\sum_{m'=-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\mathrm{d}k'_z\int\limits_0^{\infty}\mathrm{d}q'q'\left(u^*_{\vec{q}}, \alpha_{\vec{k}\vec{q}'}u_{\vec{q}'}+\beta_{\vec{k}\vec{q}'}u^*_{\vec{q}'}\right)\\
&= -\sum_{m'=-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\mathrm{d}k'_z\int\limits_0^{\infty}\mathrm{d}q'q'\left(\alpha_{\vec{k}\vec{q}'}\left(u^*_{\vec{q}}, u_{\vec{q}'}\right)+\beta_{\vec{k}\vec{q}'}\left(u^*_{\vec{q}}, u^*_{\vec{q}'}\right)\right)\\
&= \beta_{\vec{k}\vec{q}},
\end{aligned}$$

where in the last step $\left(u^*_{\vec{q}}, u^*_{\vec{q}'}\right) = -\delta_{mm'}\delta(k_z-k'_z)\frac{\delta(q-q')}{q'}$ was used.

As the modes are normalized with respect to the charge form (2.4) the Bogoliubov coefficients satisfy the relations:

$$\sum_{m=-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\mathrm{d}k_z\int\limits_0^{\infty}\mathrm{d}qq\left(\alpha_{\vec{k}\vec{q}}\alpha^*_{\vec{k}'\vec{q}}-\beta_{\vec{k}\vec{q}}\beta^*_{\vec{k}'\vec{q}}\right) = \delta^{(3)}(\vec{k}-\vec{k}') \tag{3.38}$$

$$\int \mathrm{d}^3k\left(\alpha_{\vec{k}\vec{q}}\alpha^*_{\vec{k}\vec{q}'}-\beta_{\vec{k}\vec{q}}\beta^*_{\vec{k}\vec{q}'}\right) = \delta_{mm'}\delta(k_z-k'_z)\frac{\delta(q-q')}{q'}. \tag{3.39}$$

Proof:

$$\begin{aligned}
\delta^{(3)}(\vec{k}-\vec{k}') = \left(f_{\vec{k}}, f_{\vec{k}'}\right) &= i\int \mathrm{d}\Sigma^\mu f^*_{\vec{k}}\overleftrightarrow{\partial_\mu}f_{\vec{k}'}\\
&= i\sum_{m,m'}\int \mathrm{d}k_z\mathrm{d}k'_z\int \mathrm{d}q\mathrm{d}q'qq'\int \mathrm{d}\Sigma^\mu\left(\alpha^*_{\vec{k}\vec{q}}u^*_{\vec{q}}+\beta^*_{\vec{k}\vec{q}}u_{\vec{q}}\right)\overleftrightarrow{\partial_\mu}\left(\alpha_{\vec{k}'\vec{q}'}u_{\vec{q}'}+\beta_{\vec{k}'\vec{q}'}u^*_{\vec{q}'}\right)\\
&= \sum_{m,m'}\int \mathrm{d}k_z\mathrm{d}k'_z\int \mathrm{d}q\mathrm{d}q'qq'\alpha^*_{\vec{k}\vec{q}}\alpha_{\vec{k}'\vec{q}'}\underbrace{\left(u_{\vec{q}}, u_{\vec{q}'}\right)}_{\delta_{mm'}\delta(k_z-k'_z)\delta(q-q')/q'}\\
&+\sum_{m,m'}\int \mathrm{d}k_z\mathrm{d}k'_z\int \mathrm{d}q\mathrm{d}q'qq'\beta^*_{\vec{k}\vec{q}}\alpha_{\vec{k}'\vec{q}'}\underbrace{\left(u^*_{\vec{q}}, u_{\vec{q}'}\right)}_{0}\\
&+\sum_{m,m'}\int \mathrm{d}k_z\mathrm{d}k'_z\int \mathrm{d}q\mathrm{d}q'qq'\alpha^*_{\vec{k}\vec{q}}\beta_{\vec{k}'\vec{q}'}\underbrace{\left(u_{\vec{q}}, u^*_{\vec{q}'}\right)}_{0}\\
&+\sum_{m,m'}\int \mathrm{d}k_z\mathrm{d}k'_z\int \mathrm{d}q\mathrm{d}q'qq'\beta^*_{\vec{k}\vec{q}}\beta_{\vec{k}'\vec{q}'}\underbrace{\left(u^*_{\vec{q}}, u^*_{\vec{q}'}\right)}_{-\delta_{mm'}\delta(k_z-k'_z)\delta(q-q')/q'}
\end{aligned}$$

Evaluation of the integrals with the help of the delta functions gives (3.38). The other orthogonality relation (3.39) can be shown analogously by inserting the inverse Bogoliubov transformation (3.36).

The Bogoliubov tranformations (3.35) and (3.36) imply that the annihilation operators $\hat{a}_{\vec{k}}$, $\hat{b}_{\vec{q}}$ and their adjoints $\hat{a}^\dagger_{\vec{k}}$, $\hat{b}^\dagger_{\vec{q}}$ respectively are related via[8]:

[8]This follows from the fact that the Bogoliubov transformation is a contragredient transformation.

$$\hat{b}_{\vec{q}} = \int \mathrm{d}^3k \left(\alpha_{\vec{k}\vec{q}} \hat{a}_{\vec{k}} + \beta_{\vec{k}\vec{q}} \hat{a}^{\dagger}_{\vec{k}} \right) \tag{3.40}$$

$$\hat{a}_{\vec{k}} = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}k_z \int_{0}^{\infty} \mathrm{d}qq \left(\alpha^*_{\vec{k}\vec{q}} \hat{b}_{\vec{q}} - \beta^*_{\vec{k}\vec{q}} \hat{b}^{\dagger}_{\vec{q}} \right). \tag{3.41}$$

The relations above are useful, because they tell us how $|\tilde{0}\rangle$ is connected to $|0\rangle$. We simply let $\hat{b}_{\vec{q}}$ operate on the Minkowski vacuum $|0\rangle$ to obtain:

$$\hat{b}_{\vec{q}} |0\rangle = \int \mathrm{d}^3k \left(\alpha_{\vec{k}\vec{q}} \hat{a}_{\vec{k}} + \beta_{\vec{k}\vec{q}} \hat{a}^{\dagger}_{\vec{k}} \right) |0\rangle = \int \mathrm{d}^3k \beta_{\vec{k}\vec{q}} \left|1_{\vec{k}}\right\rangle. \tag{3.42}$$

Hence we conclude that

$$\left|\tilde{0}\right\rangle = |0\rangle \Longleftrightarrow \beta_{\vec{k}\vec{q}} = 0 \quad \forall \vec{k}, \vec{q}. \tag{3.43}$$

In order to find out how the various vacua are related we need to determine the corresponding Bogoliubov coefficients. This is the subject of the following subsection.

3.7.1. Bogoliubov coefficients of cylindrical and Cartesian field modes

In the following we calculate $\alpha_{\vec{k}\vec{q}}$ and $\beta_{\vec{k}\vec{q}}$ explicitly. From (3.37) we have:

$$\begin{aligned}
\alpha_{\vec{k}'\vec{q}} &= \left(u_{\vec{q}}, f_{\vec{k}'}\right) \\
&= \frac{1}{2\pi} \frac{1}{\sqrt{2\omega}} \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_{\vec{k}'}}} i \int \mathrm{d}\Sigma^{\mu} J_m(qr) e^{-im\varphi - ik_z z} \left(e^{i\omega t} \overleftrightarrow{\partial_{\mu}} e^{-i\omega_{\vec{k}'} t} \right) e^{i\vec{k}' \cdot \vec{x}} \\
&= \frac{\omega_{\vec{k}'} + \omega}{2(2\pi)^2 \sqrt{2\pi} \sqrt{\omega} \sqrt{\omega_{\vec{k}'}}} \int \mathrm{d}z \mathrm{d}\varphi \mathrm{d}r r J_m(qr) e^{-im\varphi - ik_z z + i\vec{k}' \cdot \vec{x}(r,\varphi,z)} \\
&= \frac{\omega_{\vec{k}'} + \omega}{2(2\pi)^2 \sqrt{2\pi} \sqrt{\omega} \sqrt{\omega_{\vec{k}'}}} \int \mathrm{d}z \mathrm{d}\varphi \mathrm{d}r r e^{iz(k'_z - k_z)} e^{-im\varphi} e^{iq' r \cos(\varphi - \vartheta)} J_m(qr)
\end{aligned}$$

In the last step the definition $k'_x = q' \cos\vartheta$, $k'_y = q' \sin\vartheta$ and the standard coordinate transformation $\vec{x}(r,\varphi,z) = (r\cos\varphi, r\sin\varphi, z)$ was used. Therefore the Euclidean scalar product $\vec{k}' \cdot \vec{x}(r,\varphi,z)$ becomes $q' r \cos(\varphi - \vartheta) + k'_z z$. The z-integration gives $2\pi\delta(k'_z - k_z)$. For the factor with the cosine in the exponential we use the Jacobi-Anger expansion [Abramowitz and Stegun, 1964]

$$e^{iq' r \cos(\varphi - \vartheta)} = \sum_{n=-\infty}^{\infty} i^n J_n(q' r) e^{in(\varphi - \vartheta)}, \tag{3.44}$$

so that the integral reads:

$$\begin{aligned}
&2\pi\delta(k'_z - k_z) \sum_{n=-\infty}^{\infty} i^n e^{-in\vartheta} \underbrace{\int_0^{2\pi} \mathrm{d}\varphi e^{i\varphi(n-m)}}_{2\pi\delta_{mn}} \int_0^\infty \mathrm{d}r r J_m(qr) J_n(q'r) \\
&= (2\pi)^2 \delta(k'_z - k_z) i^m e^{-im\vartheta} \underbrace{\int_0^\infty \mathrm{d}r r J_m(qr) J_m(q'r)}_{\frac{\delta(q-q')}{q}} .
\end{aligned}$$

The factor $i^m e^{-im\vartheta}$ can be written as follows:

$$\begin{aligned}
\left(ie^{-i\vartheta}\right)^m &= \left(e^{-i(\vartheta-\frac{\pi}{2})}\right)^m \\
&= \left(\cos\left(\vartheta - \frac{\pi}{2}\right) - i\sin\left(\vartheta - \frac{\pi}{2}\right)\right)^m = \left(\sin\vartheta + i\cos\vartheta\right)^m \\
&= \left(\underbrace{\sin\arctan\left(\frac{k'_y}{k'_x}\right)}_{\frac{k'_y}{k'_x\sqrt{1+\frac{k'^2_y}{k'^2_x}}}} + i\underbrace{\cos\arctan\left(\frac{k'^2_y}{k'^2_x}\right)}_{\frac{1}{\sqrt{1+\frac{k'^2_y}{k'^2_x}}}}\right)^m = \left(\frac{k'_y + ik'_x}{\sqrt{k'^2_x + k'^2_y}}\right)^m
\end{aligned}$$

Taking all intermediate results into account we have

$$\alpha_{\vec{k}'\vec{q}} = \frac{1}{\sqrt{2\pi}} \frac{\omega_{\vec{k}'} + \omega}{2\sqrt{\omega}\sqrt{\omega_{\vec{k}'}}} \delta(k'_z - k_z) \left(\frac{k'_y + ik'_x}{\sqrt{k'^2_x + k'^2_y}}\right)^m \frac{\delta(q-q')}{q}.$$

Because of the delta functions the fraction containing ω and $\omega_{\vec{k}'}$ becomes 1[9]. Thus the final expression is:

$$\alpha_{\vec{k}'\vec{q}} = \frac{1}{2\pi} \delta(k'_z - k_z) \left(\frac{k'_y + ik'_x}{\sqrt{k'^2_x + k'^2_y}}\right)^m \frac{\delta\left(q - \sqrt{k'^2_x + k'^2_y}\right)}{q} \tag{3.45}$$

For the other coefficients $\beta_{\vec{k}'\vec{q}}$ we have:

$$\begin{aligned}
\beta_{\vec{k}'\vec{q}} &= -\left(u^*_{\vec{q}}, f_{\vec{k}'}\right) = \cdots \\
&= \frac{1}{\sqrt{2\pi}} \frac{\omega_{\vec{k}'} - \omega}{2\sqrt{\omega}\sqrt{\omega_{\vec{k}'}}} \delta(k'_z - k_z) i^m e^{-im\vartheta} \frac{\delta(q-q')}{q}.
\end{aligned}$$

But in contrast to $\alpha_{\vec{k}'\vec{q}}$ this vanishes identically

$$\beta_{\vec{k}'\vec{q}} = 0, \quad \forall \vec{k}', \vec{q}. \tag{3.46}$$

[9]Due to $\omega|_{\left(k_z = k'_z, q = \sqrt{k'^2_x + k'^2_y}\right)} = \sqrt{m_0^2 + \left(k'^2_x + k'^2_y\right)^2 + (k'_z)^2} = \omega_{\vec{k}'}$.

Hence the 'cylindrical' vacuum is equivalent to the Minkowski vacuum. This is not surprising, because of the pure spatial character of the transformation between Cartesian and cylindrical coordinates.

For accelerated observers the situation is different. Fulling [Fulling, 1973] found that a linearly accelerated observer defines naturally a vacuum state $|0\rangle_F$ different from the Minkowski vacuum $|0\rangle$. It can be shown (cf. [Birrell and Davies, 1984]) that the expectation value of the number operator $\hat{n}_{\vec{k}_F}$ for Rindler particles[10] with frequency ω_F in the Minkowski vaccum is

$$\langle 0|\, \hat{n}_{\vec{k}_F}\, |0\rangle \propto \frac{1}{e^{\frac{2\pi}{a}\omega_F} - 1},$$

where a is the constant proper acceleration of the observer. The appearance of the Planck factor $\left(e^{\frac{E}{T}} - 1\right)^{-1}$ indicates that the accelerated observer sees the Minkowski vacuum as a heat bath with temperature

$$T = \frac{a}{2\pi}\frac{\hbar}{ck_B}. \tag{3.47}$$

This is the so called Unruh effect[11]. In section 4.4 we use a detector model to confirm this result. The question as to where the energy for the creation of particles comes from arises naturally. Since energy must be supplied to keep the accelerated observer static in the Rindler frame, Unruh proposed that the force maintaining the observer's acceleration may be the origin of the creation [Unruh, 1976]. Similarly a force is needed to keep an observer onto a circular orbit. Because the coordinate system adapted to this kind of motion differs from the rotating frame only by a translation, one might expect that the rotating vacuum $|0\rangle_R$ differs from $|0\rangle$ (like $|0\rangle_F$ does).

3.7.2. The rotating vacuum

We consider the Bogoliubov transformation between the modes $\{\psi_{\vec{q}}\}$ and $\{u_{\vec{q}}\}$ to find out how the rotating vacuum $|0\rangle_R$ defined by

$$\hat{c}_{\vec{q}}\, |0\rangle_R = 0, \quad \forall \vec{q}$$

is related to the standard Minkowski (or inertial vacuum) $|0\rangle$. Hence we have to find the coefficients $\alpha_{\vec{q}\vec{q}}$, $\beta_{\vec{q}\vec{q}}$, which are given by

$$\alpha_{\vec{q}\vec{q}} = \left(u_{\vec{q}}, \psi_{\vec{q}}\right) \tag{3.48}$$

$$\beta_{\vec{q}\vec{q}} = -\left(u^*_{\vec{q}}, \psi_{\vec{q}}\right). \tag{3.49}$$

[10]The frame adapted to the accelerated observer is often called Rindler frame. By Rindler particles we understand exications of the quantum field with respect to the Fulling vacuum $|0\rangle_F$.

[11]Inserting the constants $\hbar$, c and k_B yields $T \approx 4.055 \times 10^{-21} \frac{a}{ms^{-2}} K$. Thus an acceleration of $a = 10^{21} ms^{-2}$ corresponds to a temperature of $4K$.

We perform the calculation in the inertial system and use $\bar{\varphi} = \varphi - \Omega t$:

$$\begin{aligned}\psi_{\vec{\bar{q}}}(t, r, \varphi - \Omega t, z) &= \frac{1}{2\pi} \frac{1}{\sqrt{2(\bar{\omega} + \bar{m}\Omega)}} J_{\bar{m}}(\bar{q}r) e^{-i(\bar{\omega}+\bar{m}\Omega)t} e^{i\bar{m}\varphi} e^{i\bar{k}_z z} \\ &= \left(\frac{1}{2\pi} \frac{1}{\sqrt{2\omega}} J_m(qr) e^{-i\omega t + im\varphi + ik_z z} \right) \Bigg|_{\vec{q}=\vec{\bar{q}}} = u_{\vec{\bar{q}}}(t, r, \varphi, z)\end{aligned}$$

This yields to the trivial coefficients:

$$\alpha_{\vec{\bar{q}}\vec{q}} = \left(u_{\vec{q}}, u_{\vec{\bar{q}}}\right) = \delta_{\bar{m}m} \delta(k_z - \bar{k}_z) \frac{\delta(q - \bar{q})}{\bar{q}} \tag{3.50}$$

$$\beta_{\vec{\bar{q}}\vec{q}} = -\left(u^*_{\vec{q}}, u_{\vec{\bar{q}}}\right) = 0 \tag{3.51}$$

(3.51) shows that the rotating vacuum $|0\rangle_R$ is equivalent to the standard Minkowski vacuum $|0\rangle$. The corresponding Bogoliubov transformation is just a relabeling $\vec{q} \to \vec{\bar{q}}$ of the mode functions. This is a surprising result. For the rotating observer canonical QFT predicts that the Minkowski vacuum $|0\rangle$ does not contain particles. This can be shown explicitly by calculating the expectation value of the particle number operator $\hat{n}_{\vec{\bar{q}}} = \hat{c}^\dagger_{\vec{\bar{q}}} \hat{c}_{\vec{\bar{q}}}$ with respect to $|0\rangle$. From (3.51) and (3.42) we have:

$$\langle 0| \hat{n}_{\vec{\bar{q}}} |0\rangle = \int \mathrm{d}^3 k \left|\beta_{\vec{k}\vec{\bar{q}}}\right|^2 = 0.$$

In analogy to the Unruh effect one might have expected something different.

4. Particle concept along general worldlines

In the following chapter we introduce a simple detector model and consider its response to scalar particles as it is sent on different kinds of worldlines. We show that although the detector method reproduces the Unruh effect for linearly accelerated obererververs it gives a different result for circular rotational motion. It is shown that if the field is confined inside the light cylinder the detector does not detect particles.

4.1. DeWitt detector model

In this section we want to derive the results of the previous chapter with the help of a simple detector model. The model used here goes back to DeWitt [Hawking and Israel, 1979]. The detector is an idealized point particle traveling along $x(\tau)$ and coupled via a monopole moment to a scalar field. The corresponding interaction Lagrangian is given by

$$\mathcal{L} = \lambda \hat{\mu}(\tau) \otimes \hat{\Phi}(x(\tau)), \tag{4.1}$$

where $\hat{\mu}(\tau)$ is the monopole moment operator at proper time τ of the detector, $\hat{\Phi}\,(x(\tau))$ the scalar field operator along the detector's trajectory $x(\tau)$ and λ a small coupling constant. For simplicity we suppose that the point particle has only two energy eigenstates $|E_0\rangle\,, |E_1\rangle$ with energies $E_1 > E_0$. Thus this model coincides with the model introduced in 2.3. It is assumed that the detector and the field are initially in their ground states $|E_0\rangle\,, |0\rangle$ respectively. For a general trajectory $x(\tau)$, the point particle and the field will become excited. (The energy for such an excitation can be provided by the force which keeps the detector on its trajectory). The transition amplitute A for such a process is given by [Peskin and Schroeder, 2007]:

$$A = \langle E_1| \otimes \langle \psi| \,\mathcal{T}\{e^{i\int_{-\infty}^{\infty} \mathcal{L}\mathrm{d}\tau}\}\, |E_0\rangle \otimes |0\rangle \tag{4.2}$$

where $\mathcal{T}$ indicates the time ordering symbol and $|\psi\rangle$ a general state of the quantum field $\hat{\Phi}$. Assuming that the coupling is small enough the amplitude is represented adequately by first-order pertubation theory:

$$\begin{aligned}
A &= \langle E_1| \otimes \langle\psi| \mathcal{T} \left\{ \mathbb{1} \otimes \mathbb{1} + i\lambda \int_{-\infty}^{\infty} \mathrm{d}\tau \hat{\mu}(\tau) \otimes \hat{\Phi}(x(\tau)) + \mathcal{O}(\lambda^2) \right\} |E_0\rangle \otimes |0\rangle \\
&= \langle E_1|\, E_0\rangle \langle\psi|\, 0\rangle + i\lambda \int_{-\infty}^{\infty} \mathrm{d}\tau \,\langle E_1| \hat{\mu}(\tau) |E_0\rangle \langle\psi| \hat{\Phi}(x(\tau)) |0\rangle + \mathcal{O}(\lambda^2) \\
&= i\lambda \int_{-\infty}^{\infty} \mathrm{d}\tau \,\langle E_1| \hat{\mu}(\tau) |E_0\rangle \langle\psi| \hat{\Phi}(x(\tau)) |0\rangle + \mathcal{O}(\lambda^2).
\end{aligned}$$

With the help of $\mu(\tau) = e^{i\hat{H}\tau}\hat{\mu}(0)e^{-i\hat{H}\tau}$, where $\hat{H}$ is the free-particle Hamiltonian with $\hat{H}|E_i\rangle = E_i |E_i\rangle$, $(i = 0, 1)$, A becomes

$$A = i\lambda \langle E_1| \hat{\mu}(0) |E_0\rangle \int_{-\infty}^{\infty} \mathrm{d}\tau e^{i(E_1-E_0)\tau} \langle\psi| \hat{\Phi}(x(\tau)) |0\rangle \,. \tag{4.3}$$

Next we consider the total transition probability P_A. Since the probability for the transition is given by the absolute square $|A|^2$ we sum over an orthonormal basis of field states $|\psi_n\rangle$. Using the decomposition of unity $\mathbb{1} = \sum_n |\psi_n\rangle\langle\psi_n|$ we have

$$P_A(\Delta E) = \lambda^2 |\langle E_1| \hat{\mu}(0) |E_0\rangle|^2 \int_{-\infty}^{\infty} \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\tau' e^{-i\Delta E(\tau-\tau')} \langle 0| \hat{\Phi}(x(\tau))\hat{\Phi}(x(\tau')) |0\rangle \tag{4.4}$$

with $\Delta E = E_1 - E_0$.

We restrict our attention to stationary world lines, i.e. timelike Killing trajectories, $x(\tau)$ [Letaw and Pfautsch, 1981]. For such cases the autocorrelation function $\langle 0| \hat{\Phi}(x(\tau))\hat{\Phi}(x(\tau')) |0\rangle$ is effectively a function G of the form $G(\tau - \tau')$ implying that the system is time translation invariant. (One may say that the detector is in equilibrium with the vacuum fluctuations of the scalar field). Nevertheless P_A diverges. This can be seen quickly by performing the variable substitution $s := \tau - \tau', s' := \tau + \tau'$ in the integral:

$$P_A(\Delta E) = f(E_1, E_0) \int_{-\infty}^{\infty} \mathrm{d}s' \int_{-\infty}^{\infty} \mathrm{d}s e^{-i\Delta E s} G(s), \quad f(E_1, E_0) := \frac{\lambda^2}{2} |\langle E_1| \hat{\mu}(0) |E_0\rangle|^2 \,.$$

The integral over s' gives an infinite contribution[1]. Hence we drop P_A and consider the probability per unit proper time[2] s implied by the expression above

$$R(\Delta E) := \int_{-\infty}^{\infty} \mathrm{d}s e^{-i\Delta E s} G(s), \quad G(s) = \langle 0| \hat{\Phi}\left(x(s)\right) \hat{\Phi}\left(x(0)\right) |0\rangle \tag{4.5}$$

We observe that the excitation rate (per unit proper time of the detector) is constant

[1] The function f depends on the internal structure of the detector, therefore it is not of interest for us.

[2] Another way to deal with this divergence is to consider a coupling which is adiabatically switched off as $\tau \to \pm\infty$.

with respect to s and is essentially the Fourier transform of the autocorrelation function G. Before we investigate different kinds of motion of the detector we need a suitable form for G, which is the subject of the next section.

4.2. Vacuum fluctuations of the quantized scalar field

We want to calculate the Wightman function $G(x, y) = \langle 0| \hat{\Phi}(x)\hat{\Phi}(y) |0\rangle$ for the scalar field with mass m_0 with respect to the Minkowski vacuum $|0\rangle$. For this purpose it is useful to work with the Feynman propagator $G_F(x, y)$ (see B.3) which is related to $G(x, y)$ by [Birrell and Davies, 1984]

$$G_F(x, y) = -i\Theta(x^0 - y^0)G(x, y) - i\Theta(y^0 - x^0)G(y, x). \tag{4.6}$$

In the following we assume $x^0 > y^0$ so that (4.6) reduces to

$$G_F(x, y) = -iG(x, y) = -i \langle 0| \hat{\Phi}(x)\hat{\Phi}(y) |0\rangle . \tag{4.7}$$

The Feynman propagator has the integral representation (cf. (B.14))

$$G_F(x, y) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4k \frac{e^{-ik(x-y)}}{k^2 - m_0^2 + i\epsilon}. \tag{4.8}$$

In order to evaluate this expression we make use of the following identities

$$\int_0^\infty \mathrm{d}s e^{i(\xi+i\epsilon)s} = -\frac{1}{i(\xi + i\epsilon)}, \quad \forall \xi \in \mathbb{R}$$

$$\int_{-\infty}^\infty \mathrm{d}x e^{iax^2} = \sqrt{\frac{\pi}{|a|}} e^{i\frac{\pi}{4}\frac{a}{|a|}}, \quad \forall a \in \mathbb{R} \setminus \{0\}$$

to obtain

$$G_F(x, y) = -\frac{i}{(2\pi)^4} \int_0^\infty \mathrm{d}s e^{-i(m_0^2-i\epsilon)s} \underbrace{\int \mathrm{d}^4k e^{ik^2 s - ik(x-y)}}_{\frac{\pi^2}{s^2} e^{-i\frac{\pi}{2}} e^{-i\frac{(x-y)^2}{4s}}} = -\frac{1}{16\pi^2} \int_0^\infty \frac{\mathrm{d}s}{s^2} e^{-i(m_0^2-i\epsilon)s} e^{-i\frac{(x-y)^2}{4s}} \tag{4.9}$$

In the following we drop the $i\epsilon$ prescription with the understanding that the Wightman function is to be regarded as the boundary value (on the real axis) of a function of m_0^2 and $(x - y)^2$ which is analytic in the lower half m_0^2- and $(x - y)^2$- plane.

In the case where the spacetime points x and y are timelike separated (i. e. $(x - y)^2 > 0$), it is helpful to introduce new variables

$$\zeta^2 := m_0^2(x - y)^2 > 0 \tag{4.10}$$

$$u := 2im_0^2 \frac{s}{\zeta} = 2i \frac{m_0 s}{\sqrt{(x - y)^2}}. \tag{4.11}$$

Then $G_F(x,y)$ has the following representation

$$G_F(x,y) = -\frac{i}{8\pi^2}\frac{m_0}{\sqrt{(x-y)^2}}\int\limits_{-i\infty}^{0}\mathrm{d}u\frac{e^{\frac{\zeta}{2}(u-\frac{1}{u})}}{u^2}. \tag{4.12}$$

The integration path can be deformed from the blue into the black line as shown in Figure 4.1. In this way the integral turns into the integral representation for the Hankel function $H_1^{(2)}(\zeta)$ [Arfken and Weber, 2005]

$$H_1^{(2)}(\zeta) = \frac{1}{i\pi}\int_{\mathcal{C}_H}\mathrm{d}u\frac{e^{\frac{\zeta}{2}(u-\frac{1}{u})}}{u^2}. \tag{4.13}$$

This yields

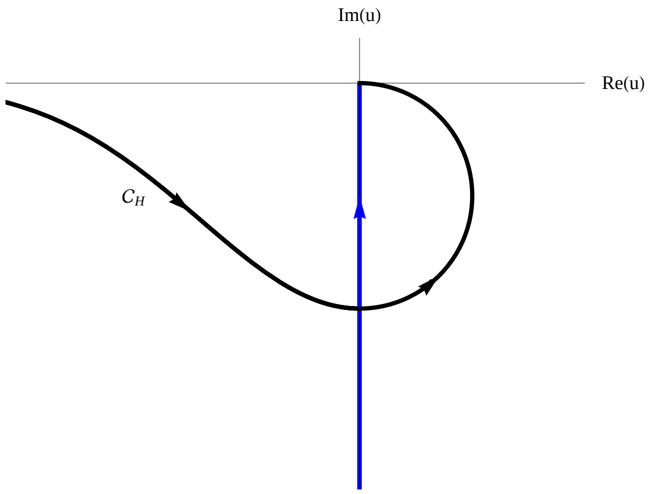

Figure 4.1.: Integration contour $\mathcal{C}_H$ for $H_1^{(2)}$.

$$G_F(x,y) = \frac{1}{8\pi}\frac{m_0}{\sqrt{(x^\mu-y^\mu)(x_\mu-y_\mu)-i\epsilon}}H_1^{(2)}\left(m_0\sqrt{(x^\mu-y^\mu)(x_\mu-y_\mu)-i\epsilon}\right). \tag{4.14}$$

Using (4.7) we obtain the final result ($x = x(s)$, $y = x(0)$)

$$R(\Delta E) = \frac{im_0}{8\pi}\int\limits_{-\infty}^{\infty}\mathrm{d}se^{-i\Delta Es}\frac{H_1^{(2)}\left(m_0\sqrt{(x^0(s)-x^0(0))^2-(\vec{x}(s)-\vec{x}(0))^2-i\epsilon}\right)}{\sqrt{(x^0(s)-x^0(0))^2-(\vec{x}(s)-\vec{x}(0))^2-i\epsilon}}. \tag{4.15}$$

This expression is difficult to evaluate for worldlines different from those corresponding to inertial motion $x(s) = (s,0,0,0)$. Therefore we consider the limit for massless particles. In this case we need the asymptotic behaviour of $H_1^{(2)}(\zeta)$ for $\zeta \to 0$. This can be found by noting that the Hankel function can be written in terms of Bessel functions [Bronstein et al., 2008]

$$H_1^{(1)}(\zeta) = J_1(\zeta) - iY_1(\zeta).$$

J_1 and Y_1 have in turn following series representations[3] [Abramowitz and Stegun, 1964]:

$$
\begin{aligned}
J_1(\zeta) &= \frac{\zeta}{2} - \frac{\zeta^3}{16} + \mathcal{O}(\zeta^5) \approx 0 \quad \text{for}\, \zeta \to 0 \\
Y_1(\zeta) &= -\frac{2}{\pi\zeta} + \zeta\frac{2\gamma - 1 - 2\log(2) + 2\log(\zeta)}{2\pi} + \mathcal{O}(\zeta^3) \approx -\frac{2}{\pi\zeta}, \quad \text{for}\, \zeta \to 0.
\end{aligned}
$$

Thus only the contribution of Y_1 is important for small arguments ζ. Taking this into account we obtain in the limit for massless particles the following two-point function:

$$G(x,y) = -\frac{1}{4\pi^2}\frac{1}{(x^\mu - y^\mu)(x_\mu - y_\mu) - i\epsilon}, \quad m_0 = 0 \tag{4.16}$$

This result is confirmed in B.4 by an independent calculation. Inserting (4.16) into (4.5) we have the following expression for the excitation rate of the detector:

$$R(\Delta E) = -\frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}s \frac{e^{-i\Delta E s}}{(x^0(s) - x^0(0) - i\epsilon)^2 - (\vec{x}(s) - \vec{x}(0))^2}. \tag{4.17}$$

In the following section we analyse this expression for three different motions of the detector.

4.3. Inertial motion and detector response

For an inertial detector the frame of reference can be chosen such that the detector is at rest. In this case the proper time s of the detector is the coordinate time t and the corresponding worldline is given by

$$x(t) = (t, 0, 0, 0).$$

Hence the excitation rate becomes

$$R(\Delta E) = -\frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}t \frac{e^{-i\Delta E t}}{(t - i\epsilon)^2}. \tag{4.18}$$

As $\Delta E > 0$ the integration can be performed in the complex t- plane by closing the contour in the lower complex t half-plane as indicated in Figure 4.2. The integration along $\mathcal{C}_2$ does not contribute due to the exponentially decreasing factor in the numerator. Therefore the integration along the real axis can be replaced by the closed contour $\mathcal{C}_1 + \mathcal{C}_2$. Because the pole at $z = i\epsilon$ is in the upper half-plane Cauchy's residue theorem implies:

$$R(\Delta E) = -\frac{1}{4\pi^2}\oint_{\mathcal{C}_1+\mathcal{C}_2} \mathrm{d}z \frac{e^{-i\Delta E z}}{(z - i\epsilon)^2} = 0. \tag{4.19}$$

[3] γ is the Euler-Mascheroni constant and is defined as the number $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right)$.

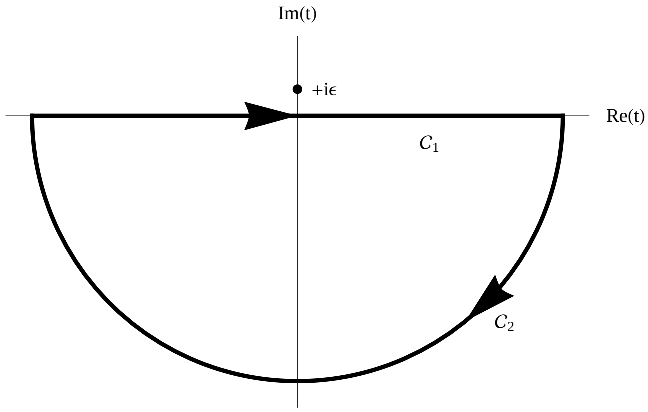

Figure 4.2.: Integration contour for the trajectory of an inertial moving detector.

This is the expected result. An inertial moving detector does not detect particles in the standard Minkowski vacuum $|0\rangle$.

4.4. Detector response for uniformly accelerated obervers

Now we want to find (4.17) for an uniformly accelerated detector. The worldline of an observer (with proper time s) who experiences a constant proper acceleration a in the x-direction is [Rindler, 2006]

$$x(s) = \left(\frac{1}{a}\sinh(as), \frac{1}{a}\cosh(as), 0, 0\right).$$

Inserting the above world line into the denominator in (4.17) gives:

$$\begin{aligned} \left(x^0(s) - x^0(0)\right)^2 &= \frac{1}{a^2}\sinh^2(as) \\ (\vec{x}(s) - \vec{x}(0))^2 &= \frac{1}{a^2}(\cosh(as) - 1)^2 = \frac{1}{a^2}(\cosh^2(as) - 2\cosh(as) + 1) \\ \Rightarrow (x^0(s) - x^0(0))^2 - (\vec{x}(s) - \vec{x}(0))^2 &= \frac{1}{a^2}(2\underbrace{\cosh(as)}_{2\sinh^2\frac{as}{2}+1} - 2) = \frac{4}{a^2}\sinh^2\frac{as}{2} \end{aligned}$$

By taking the regularization $s \to s - i\epsilon$ of the Wightman function into account the excitation rate of the accelerated detector reads:

$$R(\Delta E) = -\frac{a^2}{16\pi^2}\int_{-\infty}^{\infty} \mathrm{d}s \frac{e^{-i\Delta E s}}{\sinh^2\left(\frac{a(s-i\epsilon)}{2}\right)} = -\frac{a}{8\pi^2}\int_{-\infty}^{\infty} \mathrm{d}z \frac{e^{-i\Delta E \frac{2}{a} z}}{\sinh^2(z - i\epsilon)}.$$

In the last step an integral variable trasformation $z = as/2$ was performed.

In the following we use the formula [Prudnikov et al., 1986]

$$\frac{1}{\sinh^2 z} = \sum_{k=-\infty}^{\infty} \frac{1}{(z - i\pi k)^2}$$

to expand the integrand into a series such that

$$R(\Delta E) = -\frac{a}{8\pi^2} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}z \frac{e^{-i\Delta E \frac{2}{a} z}}{(z - ik\pi - i\epsilon)^2}. \tag{4.20}$$

As in the case of the inertial detector the integral can be evaluated by closing the contour in the lower half-plane and using Cauchy's residue theorem. The $-i\epsilon$ term shifts the pole at $z = 0$ into the upper half complex z-plane. Therefore the corresponding residue does not contribute to R. The excitation rate is effectively a sum over the residues in the lower

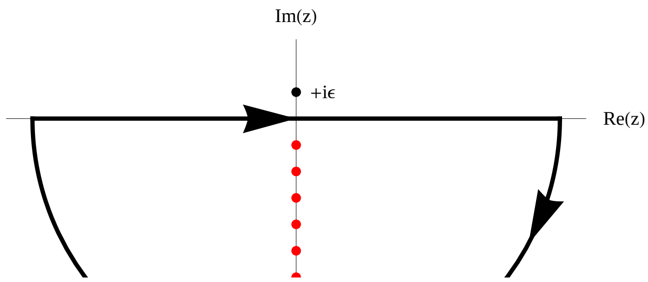

Figure 4.3.: Integration contour for the trajectory of an uniformly accelerated detector.

half-plane (red points in Figure 4.3). Using the formula for a pole of 2nd order at point $z_k = ik\pi, (k < 0)$ we have:

$$\mathrm{Res}_{z_k} = \frac{\mathrm{d}}{\mathrm{d}z}\bigg|_{z=z_k} (z - ik\pi)^2 \frac{e^{-i\Delta E \frac{2z}{a}}}{(z - ik\pi)^2} = -i\frac{2\Delta E}{a} e^{-i\Delta E \frac{2z_k}{a}} = -i\frac{2\Delta E}{a} e^{\frac{2k\pi\Delta E}{a}}$$

Hence

$$\begin{aligned} R(\Delta E) &= -2\pi i \left(-\frac{a}{8\pi^2}\right) \sum_{k=-\infty}^{1} \left(-i\frac{2\Delta E}{a} e^{k\frac{2\pi\Delta E}{a}}\right) = \frac{\Delta E}{2\pi} \sum_{k=1}^{\infty} e^{-k\frac{2\pi\Delta E}{a}} \\ &= \frac{\Delta E}{2\pi} \left(\frac{1}{1 - e^{-\frac{2\pi}{a}\Delta E}} - 1\right) = \frac{\Delta E}{2\pi} \frac{e^{-\frac{2\pi}{a}\Delta E}}{1 - e^{-\frac{2\pi}{a}\Delta E}} \end{aligned}$$

$$\Rightarrow R(\Delta E) = \frac{\Delta E}{2\pi} \frac{1}{e^{\frac{2\pi}{a}\Delta E} - 1}. \tag{4.21}$$

The second factor is the Planck factor and indicates that the detector behaves as immersed in a heat bath with temperature

$$T = \frac{a}{2\pi}.$$

So far we have seen that the detector model reproduces the results of the previous chapter for the inertial and uniformly accelerated observer. This means that the behaviour of the detector is in accordance with the frame-dependent particle concept of QFT.

4.5. Circular motion of the detector

Assuming that the detector circulates around the origin in a distance r_d with constant angular velocity $\Omega > 0$, the world line (parametrized by its proper time s) is given by

$$x(s) = (\gamma s, r_d \cos(\gamma\Omega s), r_d \sin(\gamma\Omega s), 0), \quad \gamma = \left.\frac{1}{\sqrt{1-v^2}}\right|_{v=r_d\Omega}.$$

For the denominator in the integral of (4.17) this gives

$$\begin{aligned}(x^0(s) - x^0(0) - i\epsilon)^2 - (\vec{x}(s) - \vec{x}(0))^2 &= (\gamma s - i\epsilon)^2 - (2r_d^2 - 2r_d^2\cos(\gamma\Omega s)) \\ &= (\gamma s - i\epsilon)^2 - 4r_d^2 \sin^2\frac{\gamma\Omega s}{2}.\end{aligned}$$

Performing the integration variable transformation $x = \frac{\gamma\Omega s}{2}$ the excitation rate reads:

$$R(\Delta E) = -\frac{1}{4\pi^2}\frac{\Omega}{2\gamma}\int_{-\infty}^{\infty} \mathrm{d}x \frac{e^{-i\frac{2\Delta E}{\gamma\Omega}x}}{(x - i\epsilon + v\sin x)(x - i\epsilon - v\sin x)}. \tag{4.22}$$

We want to evaluate this integral with the help of residues. The poles of the integrand are the zeros of f_+ and f_- defined by

$$\begin{aligned}f_+(z) :=& z + v\sin z \\ f_-(z) :=& z - v\sin z.\end{aligned}$$

Figure 4.4 shows the location (red points) of the poles and the choice of the integration

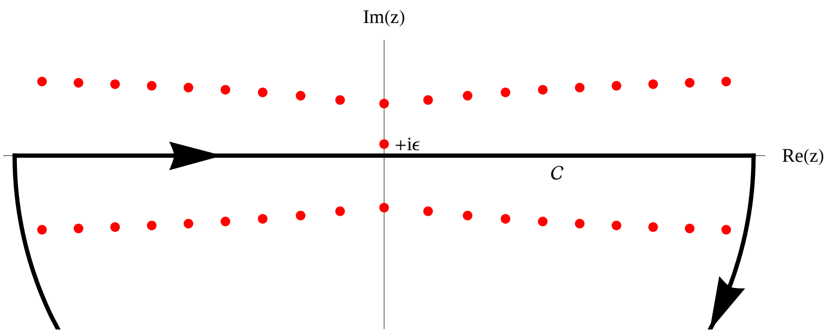

Figure 4.4.: Poles and integration contour $\mathcal{C}$.

path $\mathcal{C}$. Just as in the previous cases the contribution of the semicircle vanishes and the excitation rate is proportional to[4]

$$\oint_{\mathcal{C}} \mathrm{d}z \frac{e^{-i\frac{2\Delta E}{\gamma\Omega}z}}{f_+(z)f_-(z)}. \tag{4.23}$$

[4]We can surpress the $-i\epsilon$ term in the denominator, because the pole at $z = 0$ does not contribute for the choice of our contour.

From complex analysis we know that if a meromorphic function f has a simple pole at z and g is holomorphic in z, the residue of $\frac{g}{f}$ at z is given by $\frac{g(z)}{f'(z)}$ [Arfken and Weber, 2005]. We can use this for our problem in the following way:

- For the residues located at $z = z_+ : f_+(z_+) = 0$ we define $g(z) := \frac{e^{-i\frac{2\Delta E}{\gamma\Omega}z}}{f_-(z)}$, which is holomorphic at $z = z_+$.
- For the residues located at $z = z_- : f_-(z_-) = 0$ we define $g(z) := \frac{e^{-i\frac{2\Delta E}{\gamma\Omega}z}}{f_+(z)}$, which is holomorphic in $z = z_-$.

Thus the integral (4.23) can be evaluated in the following way:

$$\oint_C \mathrm{d}z \frac{e^{-i\frac{2\Delta E}{\gamma\Omega}z}}{f_+(z)f_-(z)} = -2\pi i \left(\sum_{z_+} \frac{e^{-i\frac{2\Delta E}{\gamma\Omega}z_+}}{f'_+(z_+)f_-(z_+)} + \sum_{z_-} \frac{e^{-i\frac{2\Delta E}{\gamma\Omega}z_-}}{f_+(z_-)f'_-(z_-)} \right).$$

With the help of this formula the excitation rate has the form

$$R(\Delta E) = \frac{i}{4\pi}\frac{\Omega}{\gamma} \left(\sum_{z_+} \frac{e^{-i\frac{2\Delta E}{\gamma\Omega}z_+}}{f'_+(z_+)f_-(z_+)} + \sum_{z_-} \frac{e^{-i\frac{2\Delta E}{\gamma\Omega}z_-}}{f_+(z_-)f'_-(z_-)} \right). \tag{4.24}$$

In contrast to the results of 3.7.2 (4.24) differs from zero. Figure 4.5 shows some numerically evaluated values of R in dependence on the energy-level splitting ΔE for different values of v. Due to the rapid convergence of (4.24) we have taken only the first 200 residues into account.

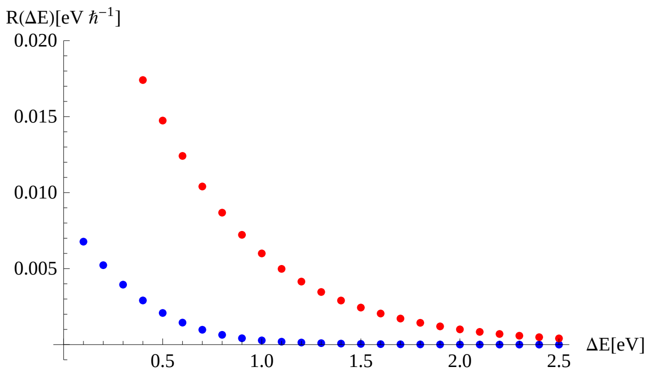

Figure 4.5.: Excitation rate for the circulating detector. Blue and red points correspond to $(\Omega = 1\frac{eV}{\hbar}, v = 0.5c)$ and $(\Omega = 1\frac{eV}{\hbar}, v = 0.75c)$ respectively.

We examine this result by considering the situation in the rotating frame of reference, where the detector is at rest and the Minkowski vacuum 'rotates'. The correct autocorrelation function needs to be evaluated with respect to $|0\rangle_R$, the natural vacuum of the rotating system (which is the same as $|0\rangle$). With the field mode decomposition (3.19) G

reads:

$$\begin{aligned}G(x(s),x(0)) &= {}_R\langle 0|\,\hat\Phi(x(s))\hat\Phi(x(0))\,|0\rangle_R\\ &= \sum_{\bar m=-\infty}^{\infty}\sum_{\bar m'=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\bar k_z\int_{-\infty}^{\infty}\mathrm{d}\bar k_z'\int_{-\infty}^{\infty}\mathrm{d}\bar q\bar q\int_{-\infty}^{\infty}\mathrm{d}\bar q'\bar q'\psi_{\vec{\bar q}}(x(s))\psi^*_{\vec{\bar q}'}(x(0))_R\langle 0|\,\hat c_{\vec{\bar q}}\hat c^\dagger_{\vec{\bar q}'}\,|0\rangle_R\\ &= \sum_{\bar m=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\bar k_z\int_{0}^{\infty}\mathrm{d}\bar q\bar q\psi_{\vec{\bar q}}(x(s))\psi^*_{\vec{\bar q}}(x(0)).\end{aligned}$$

Using the trajectory of a detector at rest in the rotating frame

$$x(s)=(\gamma s,r_d,0,0)$$

and (3.19) for the modes the excitation is given by:

$$R(\Delta E)=\frac{1}{4\pi^2}\sum_{\bar m=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\bar k_z\int_{0}^{\infty}\mathrm{d}\bar q\bar q\int_{-\infty}^{\infty}\mathrm{d}s\frac{J_{\bar m}(\bar q r_d)J_{\bar m}(\bar q r_d)}{\sqrt{m_0^2+\bar k_z^2+\bar q^2}}e^{-is(\gamma\bar\omega+\Delta E)}. \tag{4.25}$$

The s- integration yields $2\pi\delta\left(\Delta E+\gamma\sqrt{m_0^2+\bar k_z^2+\bar q^2}-\gamma\bar m\Omega\right)$. We perfom the substitution $\chi:=\sqrt{m_0^2+\bar k_z^2+\bar q^2}$, which gives:

$$\begin{aligned}\mathrm{d}\bar k_z\mathrm{d}\bar q&=\frac{\chi}{\sqrt{\chi^2-\beta^2(\bar q)}}\mathrm{d}\chi\mathrm{d}\bar q,\quad \beta^2(\bar q):=\bar q^2+m_0^2\\ \delta\left(\Delta E+\gamma\sqrt{m_0^2+\bar k_z^2+\bar q^2}-\gamma\bar m\Omega\right)&=\frac{1}{\gamma}\delta\left(\frac{\Delta E}{\gamma}+\chi-\bar m\Omega\right).\end{aligned}$$

Thus

$$R(\Delta E)=\frac{1}{\gamma}\frac{1}{2\pi}\sum_{\bar m=-\infty}^{\infty}\int_{0}^{\infty}\mathrm{d}\bar q\int_{\beta(\bar q)}^{\infty}\mathrm{d}\chi\frac{\bar q J^2_{\bar m}(\bar q r_d)}{\sqrt{\chi^2-\beta^2(q)}}\delta\left(\frac{\Delta E}{\gamma}+\chi-\bar m\Omega\right). \tag{4.26}$$

Now we note that

$$\int_B^{\infty}\mathrm{d}\chi f(\chi)\delta(A+\chi)=\Theta\left(-A-B\right)f(-A),$$

where Θ is the Heaviside step function. When applying this to the χ- integration above, we have a non vanishing contribution to R only if

$$-\left(\frac{\Delta E}{\gamma}-\bar m\Omega\right)>\beta(\bar q)=\sqrt{\bar q^2+m_0^2}. \tag{4.27}$$

Due to $\sqrt{\bar q^2+m_0^2}>0$ it follows that $\frac{\Delta E}{\gamma}-\bar m\Omega<0$. Taking the square of (4.27) we get $\left(\frac{\Delta E}{\gamma}-\bar m\Omega\right)^2>\bar q^2+m_0^2\iff\left(\frac{\Delta E}{\gamma}-\bar m\Omega\right)^2-m_0^2>\bar q^2$. The r.h.s. of the last inequality is always positive thus

$$\bar m\Omega>m_0+\frac{\Delta E}{\gamma}$$

must hold and the integration range of $\bar{q}$ is limited to

$$0 < \bar{q} < \sqrt{\left(\frac{\Delta E}{\gamma} - \bar{m}\Omega\right)^2 - m_0^2} =: A_{\bar{m}}.$$

Taking all restrictions into account we obtain

$$R(\Delta E) = \frac{1}{\gamma}\frac{1}{2\pi}\sum_{\bar{m}=\lceil\frac{\Delta E/\gamma+m_0}{\Omega}\rceil}^{\infty}\int_0^{A_{\bar{m}}}\mathrm{d}\bar{q}\frac{\bar{q}J_{\bar{m}}^2(\bar{q}r_d)}{\sqrt{A_{\bar{m}}^2-\bar{q}^2}}.$$

The integral is the same we used in 2.3 for calculating the tunneling probability. Using again [Prudnikov et al., 1983] (p.212) we get as a final result the new representation:

$$R(\Delta E) = \frac{1}{2\pi\gamma}\sum_{\bar{m}=\lceil\frac{\Delta E/\gamma+m_0}{\Omega}\rceil}^{\infty}\frac{\left(\left(-\frac{\Delta E}{\gamma}+\bar{m}\Omega\right)^2-m_0^2\right)^{\frac{2\bar{m}+1}{2}}r_d^{2\bar{m}}}{(2\bar{m}+1)!}\times$$
$$_1F_2\left(\frac{2\bar{m}+1}{2};\frac{2\bar{m}+3}{2},2\bar{m}+1;-\left(\left(-\frac{\Delta E}{\gamma}+\bar{m}\Omega\right)^2-m_0^2\right)r_d^2\right). \quad (4.28)$$

$\Delta E eV$	$R(\Delta E)\frac{eV}{\hbar}$ (4.24)	$R(\Delta E)\frac{eV}{\hbar}$ (4.28)	$R(\Delta E)\frac{eV}{\hbar}$ (4.24)	$R(\Delta E)\frac{eV}{\hbar}$ (4.28)
$2,5$	$3,18\,.10^{-9}$	$3,18\,.10^{-9}$	$1,16\,.10^{-13}$	$1,16\,.10^{-13}$
$5,0$	$1,84\,.10^{-15}$	$1,84\,.10^{-15}$	$2,16\,.10^{-38}$	$2,16\,.10^{-38}$
$7,5$	$1,12\,.10^{-21}$	$1,12\,.10^{-21}$	$3,76\,.10^{-63}$	$3,76\,.10^{-63}$
	$v = 0.1c$, $\Omega = \frac{\pi}{8}\frac{eV}{\hbar}$		$v = 0.1c$, $\Omega = \frac{\pi}{2}\frac{eV}{\hbar}$	

Table 4.1.: R evaluated for different values of Ω and $v = r_d\Omega$ for fixed ΔE. For (4.28) the first 100 and for (4.24) the residues in $-200 < \mathrm{Re}(z) < 200$, $-15 < \mathrm{Im}(z) < 0$ have been summed up.

A numerical evaluation of R according to (4.28) for $m_0 = 0$ is in agreement with (4.24) (see Table 4.1). In principle (4.28) can be used for a coupling to a massive scalar field. But a comparison with the Green function expression is not made at this point. The Hankel function appearing in (4.15) makes the numerical evaluation difficult. Summarizing we may say that the result (4.28) shows that the detector at rest in the rotating frame detects particles in the rotating vacuum $|0\rangle_R$. Although the equivalence of the vacua $|0\rangle$ and $|0\rangle_R$ (as shown in 3.7.2) might lead to the expectation that R must vanish (4.28) differs from 0. Thus the detector model is not in accordance with the approach discussed in 3.7.2.

Why is it in agreement for the linearly accelerated observer? The answer is that the accelerated detector behaves as expected, because the bath of particles in the accelerated frame is a real heat bath. Thus the detector absorbs and emits particles (which have positive energy only) as one would expect from a system being immersed in a heat bath. This can be seen as follows: Besides an excitation rate one can investigate the de-excitation rate associated with the excited detector. This rate $\tilde{R}$ is obtained by replacing ΔE in (4.20) by $-\Delta E$. This gives[5]

$$\tilde{R}(\Delta E) = \frac{\Delta E}{2\pi} \frac{1}{1 - e^{-\frac{2\pi}{a}\Delta E}}. \tag{4.29}$$

The ratio

$$\frac{R(\Delta E)}{\tilde{R}(\Delta E)} = \frac{1 - e^{-\frac{2\pi}{a}\Delta E}}{e^{\frac{2\pi}{a}\Delta E} - 1} = e^{-\frac{2\pi}{a}\Delta E} \tag{4.30}$$

has the form of a Boltzmann factor $e^{-\frac{E}{T}}$ corresponding to the Unruh temperature (3.47).

The situation for the rotating detector is different. We have seen that for this case the excitation comes from the emission of negative energy particles. In 1.3 we have shown that the presence of those particles is restricted to the ergoregion, i.e. outside the light cylinder. In the following section we investigate the situation, where the whole system (detector and scalar field) is confined to a region within the light clyinder. From the argumentation above we expect that the exciation rate R is zero in this case.

4.6. Detector in confined region

In this section we want to consider the detector moving in a confined background. More precisely we consider a real scalar field Φ restricted to a cylinder with radius R_0 and infinite height satisfying the boundary conditions:

$$\Phi|_{r=0} \text{ finite} \tag{4.31}$$

$$\Phi|_{r=R_0} = 0 \tag{4.32}$$

$$\Phi|_{\bar{\varphi}=0} = \Phi|_{\bar{\varphi}=2\pi}. \tag{4.33}$$

The apropriate modefunctions are the same as in the unconfined case with one important difference. Due to the boundary conditions (4.32) the continuous eigenvalue $\bar{q}$ gets discretized and is replaced by $\frac{\alpha_{\bar{n}\bar{m}}}{R_0}$, where $\alpha_{\bar{n}\bar{m}}$ is the $\bar{n}$th zero of $J_{\bar{m}}$. Consequently this changes the dispersion relation (2.7) to

$$\bar{\omega}_{\bar{n}} = \sqrt{\left(\frac{\alpha_{\bar{n}\bar{m}}}{R_0}\right)^2 + \bar{k}_z^2 + m_0^2} - \bar{m}\Omega, \quad \bar{n} \in \mathbb{N},\ \bar{m} \in \mathbb{Z},\ \bar{k}_z \in \mathbb{R}. \tag{4.34}$$

Hence the mode functions satisfying the boundary conditions are

$$\psi_{\vec{n}}(t, r, \bar{\varphi}, z) = N J_{\bar{m}}\left(\frac{\alpha_{\bar{n}\bar{m}}}{R_0} r\right) e^{-i\bar{\omega}_{\bar{n}} t + i\bar{m}\bar{\varphi} + i\bar{k}_z z}. \tag{4.35}$$

The appearance of the discrete eigenvalue $\frac{\alpha_{\bar{n}\bar{m}}}{R_0}$ changes the orthogonality conditon (2.9) to

$$(\psi_{\vec{n}}, \psi_{\vec{n}'}) = \delta_{\bar{m}\bar{m}'}\delta_{\bar{n}\bar{n}'}\delta(\bar{k}_z - \bar{k}_z') \tag{4.36}$$

$$(\psi_{\vec{n}}, \psi^*_{\vec{n}'}) = 0. \tag{4.37}$$

[5]For this case the integration must be closed in the upper half-plane.

We evaluate the l.h.s. of the first relation in order to fix the normalization constant N.

$$(\psi_{\vec{n}}, \psi_{\vec{n}'}) = N^* N'(\bar{\omega}_{\bar{n}} + \bar{\omega}_{\bar{n}'}) \underbrace{\int_0^{2\pi} \mathrm{d}\bar{\varphi} e^{i\bar{\varphi}(\bar{m}'-\bar{m})}}_{2\pi\delta_{\bar{m}'\bar{m}}} \underbrace{\int_{-\infty}^{\infty} \mathrm{d}z e^{iz(\bar{k}'_z-\bar{k}_z)}}_{2\pi\delta(\bar{k}'_z-\bar{k}_z)} \times$$

$$\int_0^{R_0} \mathrm{d}r r J_{\bar{m}} \left(\frac{\alpha_{\bar{n}\bar{m}}}{R_0} r\right) J_{\bar{m}'} \left(\frac{\alpha_{\bar{n}'\bar{m}'}}{R_0} r\right)$$

$$= N^* N'(\bar{\omega}_{\bar{n}} + \bar{\omega}_{\bar{n}'})(2\pi)^2 \delta_{\bar{m}'\bar{m}} \delta(\bar{k}'_z - \bar{k}_z) \underbrace{\int_0^{R_0} \mathrm{d}r r J_{\bar{m}} \left(\frac{\alpha_{\bar{n}\bar{m}}}{R_0} r\right) J_{\bar{m}} \left(\frac{\alpha_{\bar{n}'\bar{m}}}{R_0} r\right)}_{\frac{\delta_{\bar{n}\bar{n}'} R_0^2}{2} \left(J'_{\bar{m}+1}(\alpha_{\bar{n}\bar{m}})\right)^2}$$

In the last step the orthogonality condition for the Bessel functions $J_{\bar{m}}$ on a finite integral was used [Korenev, 2002] (p.96)

$$\int_0^1 \mathrm{d}r J_{\bar{m}}(\alpha_{\bar{n}\bar{m}} r) J_{\bar{m}}(\alpha_{\bar{n}'\bar{m}} r) = \frac{\delta_{\bar{n}\bar{n}'}}{2} \left(J'_{\bar{m}+1}(\alpha_{\bar{n}\bar{m}})\right)^2 .$$

Due to the Kronecker deltas $\delta_{\bar{m}\bar{m}'}$, $\delta_{\bar{n}\bar{n}'}$ and the Dirac delta $\delta(\bar{k}'_z - \bar{k}_z)$ the l.h.s. reads $(2\pi)^2 |N|^2 2\bar{\omega}_{\bar{n}} \frac{R_0}{2} \left(J'_{\bar{m}+1}(\alpha_{\bar{n}\bar{m}})\right)^2 \delta_{\bar{m}\bar{m}'} \delta_{\bar{n}\bar{n}'} \delta(\bar{k}_z - \bar{k}'_z)$. After comparison with the r.h.s. of (4.36) the normalization constant is

$$N = \frac{1}{2\pi R_0} \frac{1}{\sqrt{\bar{\omega}_{\bar{n}}}} \frac{1}{\left|J'_{\bar{m}+1}(\alpha_{\bar{n}\bar{m}})\right|} .$$

Hence the normalized mode functions in the cylinder are

$$\psi_{\vec{n}}(t, r, \bar{\varphi}, z) = \frac{1}{2\pi R_0} \frac{1}{\sqrt{\bar{\omega}_{\bar{n}}}} \frac{1}{\left|J'_{\bar{m}+1}(\alpha_{\bar{n}\bar{m}})\right|} J_{\bar{m}} \left(\frac{\alpha_{\bar{n}\bar{m}}}{R_0} r\right) e^{-i\bar{\omega}_{\bar{n}} t + i\bar{m}\bar{\varphi} + i\bar{k}_z z} \tag{4.38}$$

and the field decomposition is

$$\Phi(t, r, \bar{\varphi}, z) = \sum_{\bar{m}=-\infty}^{\infty} \sum_{\bar{n}=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\bar{k}_z \left(\psi_{\vec{n}} c_{\bar{\omega}_{\bar{n}}} + \psi^*_{\vec{n}} c^*_{\bar{\omega}_{\bar{n}}}\right) . \tag{4.39}$$

The canonical quantization procedure is straightforward (analogously to 3.5) and gives the following commutation relations for the annihilation and creation operators:

$$\left[\hat{c}_{\vec{n}}, \hat{c}^\dagger_{\vec{n}'}\right] = \delta_{\bar{n}\bar{n}'} \delta_{\bar{m}\bar{m}'} \delta(\bar{k}_z - \bar{k}'_z) \tag{4.40}$$

$$\left[\hat{c}^\dagger_{\vec{n}}, \hat{c}^\dagger_{\vec{n}'}\right] = 0 = \left[\hat{c}_{\vec{n}}, \hat{c}_{\vec{n}}\right] . \tag{4.41}$$

The vacuum state defined by the annihilators $\hat{c}_{\vec{n}}$ is identical to the standard Minkowski vaccum $|0\rangle$[6]. Thus the two point function G with respect to $|0\rangle_R$ takes the form:

$$G(x,y) = {}_R\langle 0|\,\hat{\Phi}(x)\hat{\Phi}(y)\,|0\rangle_R = \sum_{\bar{m}=-\infty}^{\infty}\sum_{\bar{n}=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\bar{k}_z \psi_{\bar{\omega}_{\bar{n}}}(x)\psi^*_{\bar{\omega}_{\bar{n}}}(y).$$

For a detector at rest in the rotating frame[7] the excitation rate (cf. (4.25)) is

$$R(\Delta E) = \frac{1}{4\pi^2 R_0^2}\sum_{\bar{m}=-\infty}^{\infty}\sum_{\bar{n}=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\bar{k}_z \frac{J^2_{\bar{m}}\left(\frac{\alpha_{\bar{n}\bar{m}}}{R_0} r_d\right)}{\left|J'_{\bar{m}+1}(\alpha_{\bar{n}\bar{m}})\right|^2\sqrt{\frac{\alpha^2_{\bar{n}\bar{m}}}{R_0^2}+\bar{k}_z^2+m_0^2}} \times \int_{-\infty}^{\infty} \mathrm{d}s e^{-is\left(\Delta E+\gamma\sqrt{\frac{\alpha^2_{\bar{n}\bar{m}}}{R_0^2}+\bar{k}_z^2+m_0^2}-\gamma\bar{m}\Omega\right)}. \tag{4.42}$$

The integration over s gives a factor $2\pi\delta\left(\Delta E+\gamma\sqrt{\frac{\alpha^2_{\bar{n}\bar{m}}}{R_0^2}+\bar{k}_z^2+m_0^2}-\gamma\bar{m}\Omega\right)$. A nonvanishing contribution to the integral is given if the argument of δ is zero. Because $\gamma, \Delta E > 0$ the detector responds under the following condition:

$$\sqrt{\frac{\alpha^2_{\bar{n}\bar{m}}}{R_0^2}+\bar{k}_z^2+m_0^2}-\bar{m}\Omega < 0. \tag{4.43}$$

Rewriting (4.43) we get $\bar{m}^2\Omega^2 > \frac{\alpha^2_{\bar{n}\bar{m}}}{R_0^2}+\bar{k}_z^2+m_0^2 \Longleftrightarrow R_0^2(\bar{m}^2\Omega^2-\bar{k}_z^2-m_0^2) > \alpha^2_{\bar{n}\bar{m}}$.

Now we use the property $\alpha_{\bar{n}\bar{m}} > \bar{m}$ of the zeros of $J_{\bar{m}}$ [Korenev, 2002] (p.95). This yields $\bar{m}^2R_0^2\Omega^2 - m_0^2 - \bar{k}_z^2 > \bar{m}^2$, which can be rearranged according to

$$R_0\Omega > \sqrt{1+\frac{m_0^2+\bar{k}_z^2}{\bar{m}^2}} \Longrightarrow R_0\Omega > 1. \tag{4.44}$$

This means the detector stays inert as long as the field is confined inside the stationary limit $r = \Omega^{-1}$. Thus our argumentation at the end of 4.5 is confirmed and answers the question raised in [Davies et al., 1996]. The detector fails to detect, if the region where the negative energy particles are located is excluded.

As R_0 increases the region where negative energy particles may exist becomes larger. On the other hand the existence of these particles allows them to be found at the position of the detector with a probability different from zero[8]. This in turn makes it possible for the detector to get excited by emitting particles with energy $\bar{\omega} = -\Delta E$.

Indeed a comparison of (2.21) with (4.28) shows that these two probabilities (i.e. the probability for finding a particle with energy $\bar{\omega} = -\Delta E$ at $r = r_d$ and the probability (per unit time) for the detector to get excited) differ only by a constant[9].

The nonvanishing excitation rate (4.28) implies that the rotating detector 'feels' some

[6]The vanishing of the Bogoliubov coefficients β is not affected by the discretization of the radial eigenvalue.
[7]Or rotating in the inertial frame.
[8]This region is forbidden for negative energy particles since the detector must be located inside the light cylinder at $r_d < \frac{1}{\Omega}$ in order to have a timelike worldline.
[9]Setting $\gamma = 1$ in (4.28) gives the excitation rate per unit coordinate time t rather than unit proper time s of the detector.

kind of Unruh radiation. In contrast to the linearly accelerated frame the spectrum of this radiation is non-thermal. This follows from the fact that the ratio $R/\tilde{R}$ (where $\tilde{R}$ is the de-excitation rate obtained by replacing $\Delta E \to -\Delta E$ in (4.22) or (4.25)) has not the form of a Boltzmann factor. Nevertheless one is able to assign an effective temperature T_{eff} by demanding that $R(\Delta E)/\tilde{R}(\Delta E) = \exp(-\Delta E/T_{eff})$, so that

$$T_{eff}(\Delta E) = \Delta E \log \left(\frac{\tilde{R}(\Delta E)}{R(\Delta E)} \right). \tag{4.45}$$

Bell and Leinaas suggested that (4.45) could be used as a measure for the Unruh radiation in the rotating frame and that this may be verified experimentally with the help of electrons in storage rings. More precisely they proposed in [Bell and Leinaas, 1983] that the effective temperature (4.45) could be related to the (experimentally verified) depolarization of electrons in magnetic fields[10]. But in a second paper they raised doubts about their idea [Bell and Leinaas, 1987]. More recent work [Unruh, 1998] showed that it is too naive to expect that a thermal approximation of the Unruh radiation in rotating frames is sufficient to describe the residual spin polarization of the electrons.

[10]This depolarization, called Sokolov-Ternov effect, is due to the synchrotron radiation coming from a spin flip transition of the electrons [Sokolov and Ternov, 1964].

5. Conclusions

The aim of the present thesis was to intepret the excitation of the detector as a tunneling of particles with negative energy. In this context the tunneling picture (RQM) was compared with the detector model (QFT).

- **Tunneling Method**

  The tunneling probability Γ of negative energy particles in the rotating frame was investigated (see 2.3). Thereby the probability of a particle to propagate from the classical allowed ergoregion $r > \Omega^{-1}$ to the location of the detector r_d was considered. In order to find an appropriate expression for this probability the exact radial wave function

  $$\phi(r) = J_{\bar{m}}\left(\frac{Q}{\hbar}r\right)$$

  and its semiclassical WKB approximation

  $$\phi_1(r) = \sqrt{\frac{\hbar}{2\pi}}\frac{1}{\sqrt[4]{\bar{L}_z^2 - r^2\bar{Q}^2}}\left(\frac{\bar{L}_z + \sqrt{\bar{L}_z^2 - \bar{Q}^2 r^2}}{\bar{Q}r}\right)^{-\bar{m}} e^{+\frac{1}{\hbar}\sqrt{\bar{L}_z^2 - \bar{Q}^2 r^2}}$$

  were examined.

- **Detector model**

  The DeWitt detector model (cf. 4.1) was used to show that a rotating observer detects particles in the Minkowski vacuum. The corresponding excitation rate R in both the inertial as well as the rotating frame was calculated. First for a detector in the background of a unconfined scalar field 4.5 and second for a field confined in a cylinder 4.6.

The results can be summarized as follows:

(i) The exact quantum mechanical tunneling probability Γ is not equivalent to the excitation rate R found with methods of QFT:

$$\Gamma(\Delta E) = \sum_{\bar{m}=\lceil\frac{\Delta E + m_0}{\Omega}\rceil}^{\infty} \frac{\left((-\Delta E + \bar{m}\Omega)^2 - m_0^2\right)^{\frac{2\bar{m}+1}{2}} r_d^{2\bar{m}}}{J_{\bar{m}}^2(\bar{m})(2\bar{m}+1)!} \times {}_1F_2\left(\frac{2\bar{m}+1}{2}; \frac{2\bar{m}+3}{2}, 2\bar{m}+1; -((-\Delta E + \bar{m}\Omega)^2 - m_0^2)r_d^2\right) \quad (5.1)$$

$$R(\Delta E) = \frac{1}{2\pi\gamma} \sum_{\bar{m}=\lceil \frac{\Delta E/\gamma + m_0}{\Omega} \rceil}^{\infty} \frac{\left(\left(-\frac{\Delta E}{\gamma} + \bar{m}\Omega\right)^2 - m_0^2\right)^{\frac{2\bar{m}+1}{2}} r_d^{2\bar{m}}}{(2\bar{m}+1)!} \times$$
$$ {}_1F_2\left(\frac{2\bar{m}+1}{2}; \frac{2\bar{m}+3}{2}, 2\bar{m}+1; -\left(\left(-\frac{\Delta E}{\gamma} + \bar{m}\Omega\right)^2 - m_0^2\right) r_d^2\right). \quad (5.2)$$

(ii) The semiclassical tunneling probability Γ_{WKB} does not match R either (cf. Appendix A.5).

(iii) The excitation rate R of the detector is the same in the inertial and the rotating frame. This was shown numerically in Table 4.1.

(iv) The excitation rate R was found to be proportional to the total charge density ρ of a negative energy particle at the position of the detector (see (2.21) and (4.28)).

(v) It was shown that the detector stays inert as long as space is confined within the light cylinder ($R_0 < \Omega^{-1}$). In this case the absence of negative energy field modes leads to a vanishing excitation rate R (cf. 4.44).

A. Appendix A

A.1. WKB approximation of the radial Klein-Gordon wavefunction

The aim of this section is to find a semiclassical expression of the radial Klein-Gordon wavefunction with the help of the WKB method. This approximation method uses $\hbar$ as semiclassical parameter so that the wavefunction is expanded in powers of $\hbar$. Thus we consider the Klein-Gordon equation with units such that $c = 1$, but $\hbar$ appears explicitly and being treated as an expansion parameter. This change of units has the effect that $m_0 \to \frac{m_0}{\hbar}$ and

$$\bar{q}^2 = (\bar{\omega} + \bar{m}\Omega)^2 - \bar{k}_z^2 - \frac{m_0^2}{\hbar^2} \Longrightarrow \hbar^2 \bar{q}^2 = (\bar{E} + \bar{L}_z\Omega)^2 - \bar{P}_z^2 - m_0^2 =: \bar{Q}^2.$$

Here we have used the well-known relations $\bar{E} = \hbar\bar{\omega}$, $\bar{L}_z = \hbar\bar{m}$, $\bar{P}_z = \hbar\bar{k}_z$ between energy, angular momentum, spatial 3-momentum and their corresponding quantum numbers. As shown in chapter 2 the exact radial wavefunction ϕ is given by the Bessel function $\phi(r) = J_{\bar{m}}(\bar{q}r) = J_{\bar{m}}\left(\frac{\bar{Q}}{\hbar}r\right)$ and satisfies the differential equation [Korenev, 2002]

$$\phi''(r) + \frac{1}{r}\phi'(r) + \frac{1}{\hbar^2}\left(\bar{Q}^2 - \frac{\bar{L}_z^2}{r^2}\right)\phi(r) = 0. \tag{A.1}$$

The WKB method is best applicable to ODEs of the form $u''(r) \pm \frac{1}{\hbar^2}\kappa(r)u(r) = 0$. Here the function κ is nearly constant in the validity region of the WKB expansion. Setting

$$\phi(r) = \frac{u(r)}{\sqrt{r}}$$

the radial Klein-Gordon equation reads:

$$u''(r) + \frac{1}{\hbar^2}\left(\bar{Q}^2 - \frac{\bar{L}_z^2 - \frac{\hbar^2}{4}}{r^2}\right)u(r) = 0.$$

It is useful to introduce a parameter $\lambda^2 := \bar{L}_z^2 - \frac{\hbar^2}{4}$ and to define $\kappa_\lambda(r) := \sqrt{\bar{Q}^2 - \frac{\lambda^2}{r^2}}$. Thus we obtain the following ODE for u:

$$u''(r) + \frac{1}{\hbar^2}\kappa_\lambda(r)^2 u(r) = 0. \tag{A.2}$$

The ansatz

$$u(r) = \exp \frac{i}{\hbar} S(r), \tag{A.3}$$

implies an ODE for the phase S of the wavefunction u:

$$-\frac{\hbar}{i} S''(r) - S'^2(r) + \kappa_\lambda^2(r) = 0. \tag{A.4}$$

The WKB approximation starts with the expansion of S in a power series of $\hbar$ [Dunham, 1932][1]:

$$S(r) = \sum_{n=0}^{\infty} \left(\frac{\hbar}{i}\right)^n S_n(r) \tag{A.5}$$

$$\Rightarrow S'(r) = \sum_{n=0}^{\infty} \left(\frac{\hbar}{i}\right)^n S_n'(r), \quad S''(r) = \sum_{n=0}^{\infty} \left(\frac{\hbar}{i}\right)^n S_n''(r).$$

Cauchy's product formula for infinite series [Meyberg and Vachenauer, 2001]

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{m=0}^{\infty} b_m\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)$$

applied to S'^2 gives

$$S'^2(r) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left(\frac{\hbar}{i}\right)^n S_k'(r) S_{n-k}'(r)\right).$$

Putting this into (A.4) yields

$$\underbrace{\sum_{n=0}^{\infty}\left\{-\left(\frac{\hbar}{i}\right)^{n+1} S_n''(r) - \left(\frac{\hbar}{i}\right)^n \sum_{k=0}^{\infty} S_k'(r) S_{n-k}'(r)\right\}}_{-\sum_{n=1}^{\infty}\left(\frac{\hbar}{i}\right)^n\left(S_{n-1}''(r)+\sum_{k=0}^{n} S_k'(r)S_{n-k}'(r)\right)-S_0'^2(r)} + \kappa_\lambda^2(r) = 0$$

The l.h.s. is zero iff the coefficients in front of each power of $\hbar$ vanish. Taking into account that $\kappa_\lambda \in \mathcal{O}(\hbar^0)$ we obtain the following set of equations:

$$-S_0'^2(r) + \kappa_\lambda^2(r) = 0 \tag{A.6}$$

$$S_{n-1}''(r) + \sum_{k=0}^{n} S_k'(r) S_{n-k}'(r) = 0 \tag{A.7}$$

These equations allow us to calculate arbitrary orders of the WKB expansion (A.5) and to obtain the WKB-solutions u_n, defined by

$$u_n(r) := C_n \exp\left(\frac{i}{\hbar}\sum_{k=0}^{n}\left(\frac{\hbar}{i}\right)^k S_k(r)\right), \tag{A.8}$$

[1] More about the convergence of the WKB series can be found in [Kevorkian and Cole, 1996].

where C_n is the normalization constant.

We integrate (A.6)

$$S_0(r) = \pm \int_{r_\lambda}^{r} \mathrm{d}r' \sqrt{\bar{Q}^2 - \frac{\lambda^2}{r'^2}} = \pm \left(\sqrt{\bar{Q}^2 r^2 - \lambda^2} - \lambda \arccos \frac{\lambda}{\bar{Q} r} \right), \quad r > r_\lambda := \frac{\lambda}{\bar{Q}}, \tag{A.9}$$

to obtain the lowest WKB approximation of the radial wavefunction

$$u_0(r) = C_0^{\pm} \exp \left\{ \pm \frac{i}{\hbar} \left(\sqrt{\bar{Q}^2 r^2 - \lambda^2} - \lambda \arccos \frac{\lambda}{\bar{Q} r} \right) \right\}. \tag{A.10}$$

The $+(-)$ sign corresponds to a radial outgoing (incoming) wave. In the region $r < r_\lambda$, where the phase S_0 becomes imaginary, the wave function u_0 becomes real. Thus u_0 has an exponentially decreasing (increasing) behaviour for $r \approx 0$ depending on which sign of κ_λ in (A.6) is chosen. The explicit form of S_0 for $r < r_\lambda$ can be obtained by using the definition of the inverse cosine in terms of complex logarithms [Abramowitz and Stegun, 1964]:

$$\begin{aligned} \arccos \frac{\lambda}{\bar{Q} r} &= \frac{\pi}{2} + i \ln \left(i \frac{\lambda}{\bar{Q} r} + \sqrt{1 - \frac{\lambda^2}{\bar{Q}^2 r^2}} \right) = \frac{\pi}{2} + i \ln \left(i \frac{\lambda + \sqrt{\lambda^2 - \bar{Q}^2 r^2}}{\bar{Q} r} \right) \\ &= \frac{\pi}{2} + i \ln i + i \ln \left(\frac{\lambda + \sqrt{\lambda^2 - \bar{Q}^2 r^2}}{\bar{Q} r} \right) = i \ln \left(\frac{\lambda + \sqrt{\lambda^2 - \bar{Q}^2 r^2}}{\bar{Q} r} \right). \end{aligned}$$

Thus in the region $r < r_\lambda$ the WKB solution u_0 has the following form:

$$u_0(r) = C_0'^{\pm} \left(\frac{\lambda + \sqrt{\lambda^2 - \bar{Q}^2 r^2}}{\bar{Q} r} \right)^{\mp \frac{\lambda}{\hbar}} e^{\pm \frac{1}{\hbar} \sqrt{\lambda^2 - \bar{Q}^2 r^2}}, \quad r < r_\lambda \tag{A.11}$$

In order to obtain the decreasing behaviour of the semiclassical wave function for large r we take the next higher order of (A.5) into account. Putting $n = 1$ in (A.7) we get

$$S_1'(r) = -\frac{1}{2} \frac{S_0''(r)}{S_0'(r)} \Longrightarrow S_1(r) = -\frac{1}{2} \ln |S_0'(r)| + const. \tag{A.12}$$

Absorbing the integration constant into C_1 (respectively C_1') we get for the first order WKB approximation

$$u_1(r) = \frac{C_1}{\sqrt[4]{\bar{Q}^2 - \frac{\lambda^2}{r^2}}} \exp \left\{ \pm \frac{i}{\hbar} \left(\sqrt{\bar{Q}^2 r^2 - \lambda^2} - \lambda \arccos \frac{\lambda}{\bar{Q} r} \right) \right\}, \quad r > r_\lambda \tag{A.13}$$

$$u_1(r) = \frac{C_1'}{\sqrt[4]{\frac{\lambda^2}{r^2} - \bar{Q}^2}} \left(\frac{\lambda + \sqrt{\lambda^2 - \bar{Q}^2 r^2}}{\bar{Q} r} \right)^{\mp \frac{\lambda}{\hbar}} e^{\pm \frac{1}{\hbar} \sqrt{\lambda^2 - \bar{Q}^2 r^2}}, \quad r < r_\lambda. \tag{A.14}$$

A.1.1. The validity of the WKB-solutions

The expansion (A.5) can be turncated at order n if $|S_n| >> \hbar\,|S_{n+1}|$. For $n = 0$ the condition is $|S_0| >> \hbar|S_1|$. For $n = 2$ the condition $|S_1| >> \hbar|S_2|$ may be rewritten into $\hbar|S_0'| >> |S_1'|$ (for more details see [Nolting, 2006] (p. 198)). Thus we expect that (A.8) turncated at $n = 1$ gives a good approximation to u if

$$\hbar\left|\frac{S_1'}{S_0'}\right| << 1.$$

Using the equations (A.6) and (A.12) the validity condition for the WKB solutions can be written as

$$\frac{\hbar}{2}\left|\frac{S_0''}{S_0'^2}\right| = \frac{\hbar}{2}\left|\frac{\kappa_\lambda'}{\kappa_\lambda^2}\right| = \frac{\hbar}{2}\frac{\lambda^2}{r^3}\left|\bar{Q}^2 - \frac{\lambda^2}{r^2}\right|^{-\frac{3}{2}} << 1. \tag{A.15}$$

This is fullfilled if:

- $r >> \frac{\lambda}{\bar{Q}}$ and $r >> \bar{Q}(\hbar\frac{\lambda^2}{2})^{\frac{1}{3}}$, or
- $\frac{\hbar}{2\lambda} << 1$ and $r << \frac{\lambda}{\bar{Q}}$.

A.2. Langer correction

In the following we want to modify the parameter λ in the WKB approximation. The idea is to make use of the second validity condition above. We compare the ODE for u with the ODE satisfied by u_n in the vicinity of $r \approx 0$. This will yield an algebraic equation for λ in terms of $\bar{m}$. Near the origin (A.2) behaves according to:

$$u''(r) - \left(\bar{m}^2 - \frac{1}{4}\right)\frac{1}{r^2}u(r) = 0. \tag{A.16}$$

A.2.1. Zeroth WKB order

The zeroth order WKB solution satisfies

$$u_0''(r) - \left(\frac{i}{\hbar}S_0''(r) - \frac{1}{\hbar^2}S_0'^2(r)\right)u_0(r) = 0. \tag{A.17}$$

(A.6) implies that $S_0'(r) = \pm\sqrt{\bar{Q}^2 - \frac{\lambda^2}{r^2}}$ and $S_0''(r) = \mp\frac{\lambda^2}{r^3}\frac{1}{\sqrt{\bar{Q}^2 - \frac{\lambda^2}{r^2}}}$. For $r \approx 0$ we have $\bar{Q}^2 - \frac{\lambda^2}{r^2} \approx -\frac{\lambda^2}{r^2}$. Thus the asymptotic form of (A.17) for $r \approx 0$ is

$$u_0''(r) - \left(\mp\frac{1}{\hbar}\lambda + \frac{1}{\hbar^2}\lambda^2\right)\frac{1}{r^2}u_0(r) = 0. \tag{A.18}$$

In order to match u_0 with u near the origin the parameter λ needs to be a solution of

$$\mp\frac{1}{\hbar}\lambda + \frac{1}{\hbar^2}\lambda^2 = \bar{m}^2 - \frac{1}{4}. \tag{A.19}$$

Thus for $u_0(r) = C_0 \exp\left(\pm\frac{i}{\hbar}\int^r \kappa_\lambda(r')\mathrm{d}r'\right)$ we obtain

$$\lambda = \bar{L}_z \pm \frac{\hbar}{2}. \tag{A.20}$$

Where the upper sign[2] corresponds to the upper sign in the exponential of u_0.

A.2.2. First WKB order

The next higher order in (A.8) is $u_1 = C_1 \exp\left(\frac{i}{\hbar}S_0 + S_1\right)$. For the corresponding differential equation we need its second derivative, which is

$$\begin{aligned}
u_1'' &= \left(-\frac{1}{\hbar^2}S_0'^2 + \underbrace{S_1'^2}_{=\frac{S_0''^2}{4S_0'^2}} + \frac{i}{\hbar}\underbrace{(2S_0'S_1' + S_0'')}_{=0} + \underbrace{S_1''}_{-\frac{S_0'''}{2S_0'}+\frac{S_0''^2}{2S_0'^2}}\right)u_1. \\
\Longrightarrow u_1'' &- \left(-\frac{1}{\hbar^2}S_0'^2 + \frac{3}{4}\frac{S_0''^2}{S_0'^2} - \frac{1}{2}\frac{S_0'''}{S_0'}\right)u_1 = 0 \\
\Longrightarrow u_1'' &- \left(-\frac{1}{\hbar^2}\left(\bar{Q}^2 - \frac{\lambda^2}{r^2}\right) + \frac{3}{4}\frac{\lambda^4}{r^6}\frac{1}{\left(\bar{Q}^2 - \frac{\lambda^2}{r^2}\right)^2} + \frac{1}{2}\frac{\lambda^4}{r^6}\frac{1}{\left(\bar{Q}^2 - \frac{\lambda^2}{r^2}\right)^2} + \frac{3}{2}\frac{\lambda^2}{r^4}\frac{1}{\bar{Q}^2 - \frac{\lambda^2}{r^2}}\right)u_1 = 0 \\
\Longrightarrow u_1''(r) &- \left(\frac{1}{\hbar^2}\lambda^2 + \frac{5}{4} - \frac{3}{2}\right)\frac{1}{r^2}u_1(r) = 0, \quad r \approx 0
\end{aligned}$$

A comparison of the last equation with (A.16) yields

$$\lambda^2 = \bar{L}_z^2. \tag{A.21}$$

Hence we see that the replacement $\bar{m}^2 - \frac{1}{4} \to \bar{m}^2$ improves the accuracy of the first order WKB-solution.

The necessity of the modification of the centrifugal barrier term to improve the accuracy of the first order WKB approximation was mentioned first by [Kramers, 1926] and proven by Langer [Langer, 1937]. Therefore (A.21) is often called Langer correction. It must be expected that for higher orders a different choice of λ is necessary. Indeed [Beckel and Nakhleh, 1963] showed that for u_2 (A.21) is no longer justified. How the centrifugal term must be chosen for arbitary u_n can be found in [Vasan and Seetharaman, 1984].

A.3. Connection with classical mechanics

The WKB approximation gives an elegant way to check Bohr's correspondence principle (Bohr, 1920), which states that for large quantum numbers and $\hbar \to 0$, QM reproduces

[2] Equation (A.19) gives a second solution $\lambda = \hbar\left(-\bar{m} \pm \frac{1}{2}\right)$. In the following we choose the positive definite solution, since the corresponding WKB solution gives a better approximation to the exact wavefunction.

classical mechanics[3].

Now we consider the WKB-ansatz (A.3) in the limit $\hbar \to 0$. In this case the semiclassical radial wavefunction is dominated by the zeroth WKB order u_0, which in turn is determined by

$$S_0'^2(r) = \bar{Q}^2 - \left.\frac{\lambda^2}{r^2}\right|_{\lambda=\bar{L}_z \pm \frac{\hbar}{2}} \to \bar{Q}^2 - \frac{\bar{L}_z^2}{r^2}, \quad \hbar \to 0. \tag{A.22}$$

In the following we show that this is the Hamilton-Jacobi equation[4] (HJE) of the free particle in the rotating frame. The HJE can be obtained from the classical Hamiltonian by replacing the momenta p_μ by $\frac{\partial S}{\partial x^\mu}$. S denotes the classical action of the particle [Nolting, 2010]. Applying these rules to the Hamiltonian of the free particle in the rotating frame (1.27) we obtain following partial differential equation[5]:

$$\frac{\partial S}{\partial t} = \sqrt{m_0^2 + \left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \bar{\varphi}}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2} - \Omega \frac{\partial S}{\partial \bar{\varphi}}. \tag{A.23}$$

Due to the cyclicity of t, $\bar{\varphi}$ and z the following ansatz is allowed [Nolting, 2010]:

$$S = \bar{E}t + \bar{L}_z \bar{\varphi} + \bar{P}_z z + S_0(r). \tag{A.24}$$

Thus (A.23) reads

$$\bar{E} = \sqrt{m_0^2 + \left(\frac{\partial S_0}{\partial r}\right)^2 + \frac{\bar{L}_z^2}{r^2} + \bar{P}_z^2} - \Omega \bar{L}_z \tag{A.25}$$

$$\Longleftrightarrow \left(\frac{\partial S_0}{\partial r}\right)^2 = \left(\bar{E} + \Omega \bar{L}_z\right)^2 - \bar{P}_z^2 - m_0^2 - \frac{\bar{L}_z^2}{r^2} = \bar{Q}^2 - \frac{\bar{L}_z^2}{r^2} \tag{A.26}$$

and shows that indeed the phase of the lowest WKB-order satisfies the HJE.

A.4. Comparison with exact solution

Essentially u_1 can be written as

$$u_1(r) = \frac{C^\pm}{\sqrt{\kappa_{\bar{m}}(r)}} \exp\left\{ \pm \frac{i}{\hbar} \int\limits_{r_L}^{r} \mathrm{d}t \kappa_{\bar{m}}(t) \right\}. \tag{A.27}$$

[3]More precisely the limit means that as $\hbar \to 0$ and $\bar{\omega}$, $\bar{m}$, $\bar{k}_z \to \infty$ the products $\bar{E} = \hbar\bar{\omega}$, $\bar{L}_z = \hbar\bar{m}$, $\bar{P}_z = \hbar\bar{k}_z$ stay finite.

[4]The HJE provides an equivalent description of a classical system which is determined by the Hamilton equations.

[5]Note that $H = p_0$.

Where r_L indicates the classical turning point of the point particle (1.16). The general WKB wavefunction is a linear combination of outgoing and incoming solutions:

$$u_1(r) = \frac{C^+}{\sqrt{\kappa_{\bar{m}}(r)}} \exp\left\{+\frac{i}{\hbar}\int_{r_L}^{r} \mathrm{d}t\kappa_{\bar{m}}(t)\right\} + \frac{C^-}{\sqrt{\kappa_{\bar{m}}(r)}} \exp\left\{-\frac{i}{\hbar}\int_{r_L}^{r} \mathrm{d}t\kappa_{\bar{m}}(t)\right\}. \tag{A.28}$$

To fix the coefficients $C^\pm$ we compare u_1 with the exact solution $u = \sqrt{r}J_{\bar{m}}\left(\frac{\bar{Q}}{\hbar}r\right)$ in the asymptotic region $r \to \infty$. Here we expect that the WKB soultion gives a good approximation (cf. (A.15)). A comparison between the asymptotic behaviour of the Bessel functions [Bronstein et al., 2008]

$$\sqrt{r}J_{\bar{m}}\left(\frac{\bar{Q}}{\hbar}r\right) \approx \sqrt{\frac{2\hbar}{\pi\bar{Q}}}\cos\left(\frac{\bar{Q}r}{\hbar} - \frac{\bar{m}\pi}{2} - \frac{\pi}{4}\right), \quad r \approx \infty \tag{A.29}$$

and (A.13)

$$u_1(r) \approx \frac{1}{\sqrt{\bar{Q}}}\left(C^+ e^{\frac{i\bar{Q}r}{\hbar} - \frac{i\bar{m}\pi}{2}} + C^- e^{-\frac{i\bar{Q}r}{\hbar} + \frac{i\bar{m}\pi}{2}}\right), \quad r \approx \infty \tag{A.30}$$

shows that $C^+ = \sqrt{\frac{2\hbar}{\pi}}\frac{1}{2}e^{-i\frac{\pi}{4}} = (C^-)^*$. Therefore the first order approximation is

$$u_1(r) = \sqrt{\frac{2\hbar}{\pi}}\frac{1}{\sqrt[4]{\bar{Q}^2 - \frac{\bar{L}_z^2}{r^2}}}\cos\left(\frac{1}{\hbar}\sqrt{\bar{Q}^2r^2 - \bar{L}_z^2} - \bar{m}\arccos\left(\frac{\bar{L}_z}{\bar{Q}r}\right) - \frac{\pi}{4}\right), \; r > r_L. \tag{A.31}$$

To get the coefficients $C'^\pm$ of the solution in the classically forbidden region we make use of[6] [Bronstein et al., 2008]

$$\sqrt{r}J_{\bar{m}}\left(\frac{\bar{Q}}{\hbar}r\right) \approx \sqrt{r}\frac{\left(\frac{\bar{Q}r}{\hbar}\right)^{\bar{m}}}{2^{\bar{m}}\bar{m}!} \approx \frac{r^{\bar{m}+\frac{1}{2}}\left(\frac{\bar{Q}}{2\hbar}\right)^{\bar{m}}}{\sqrt{2\pi}\sqrt{\bar{m}}}e^{\bar{m}-\bar{m}\ln\bar{m}}, \quad r \approx 0, \; \bar{m} \to \infty \tag{A.32}$$

and compare it with the asymptotic form of the WKB solution u_1 (A.14) near the origin:

$$u_1(r) \approx C'^+\sqrt{\frac{r}{\bar{L}_z}}\left(\frac{\bar{Q}r}{2\hbar}\right)^{\bar{m}}e^{\bar{m}-\bar{m}\ln\bar{m}} + C'^-\sqrt{\frac{r}{\bar{L}_z}}\left(\frac{\bar{Q}r}{2\hbar}\right)^{-\bar{m}}e^{-\bar{m}+\bar{m}\ln\bar{m}}. \tag{A.33}$$

We see that in order to match u_1 with u at $r \approx 0 \, (\bar{m} \to \infty)$ the coefficients C'^+ and C'^- must be $\sqrt{\frac{\hbar}{2\pi}}$ and 0 respectively. Thus we have[7]

$$u_1(r) = \sqrt{\frac{\hbar}{2\pi}}\frac{1}{\sqrt[4]{\frac{\bar{L}_z^2}{r^2} - \bar{Q}^2}}\left(\frac{\bar{L}_z + \sqrt{\bar{L}_z^2 - \bar{Q}^2r^2}}{\bar{Q}r}\right)^{-\bar{m}}e^{+\frac{1}{\hbar}\sqrt{\bar{L}_z^2-\bar{Q}^2r^2}}, \quad r < r_L. \tag{A.34}$$

Figure A.1 is a plot of the exact radial wavefunction u (thick red line) and its first two WKB approximations u_0, u_1 (grey and blue line, respectively) for the values $\bar{Q} = 6\frac{eV}{c}$ and $\bar{L}_z = 3\hbar$.

[6] In the last step of (A.32) we used Stirling's formula for large values of the factorial function $\bar{m}! \approx \sqrt{2\pi\bar{m}}e^{-\bar{m}+\bar{m}\ln\bar{m}}$.

[7] (A.34) shows that a simple analytic continuation of (A.31) into the forbidden region gives not the correct wave function.

It shows that u_1 becomes a good approximation as one approaches the asymptotic regions $r << r_L$ and $r >> r_L$ but fails at $r \approx r_L$. This is in accordance with the validity conditions of the WKB approximation (A.15).

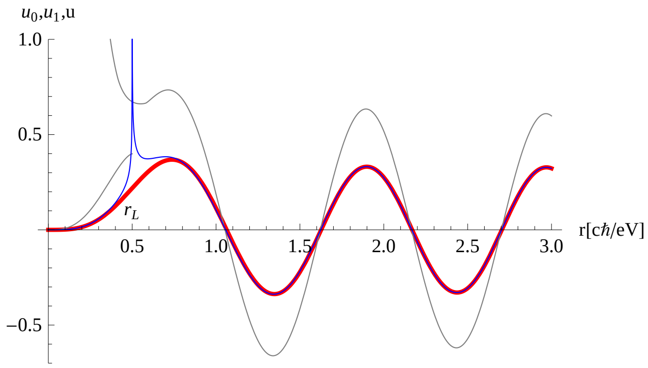

Figure A.1.: WKB Approximation of radial wavefunction.

A.5. WKB approximation and quantum tunneling

The WKB method can be applied very successfully to the quantum mechanical phenomenon of tunneling of particles through classically forbidden regions. Figure A.2 shows a typical potential barrier[8], drawn as a function of the radial coordinate.

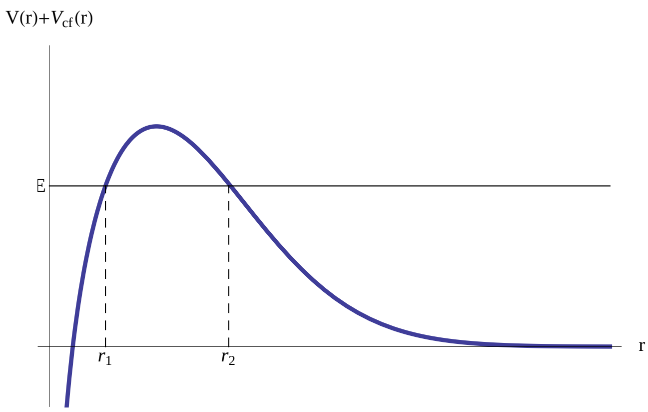

Figure A.2.: Potential barrier with classically forbidden region $r_1 < r < r_2$.

The corresponding semiclassical transmission cofficient [Brack and Bhaduri, 2003]

$$\gamma_{WKB}(E) \approx e^{-\frac{2}{\hbar}\mathrm{Im}\left(\int\limits_{r_1}^{r_2} \mathrm{d}r\sqrt{(E-V(r))^2-m_0^2-V_{cf}(r)}\right)}, \tag{A.35}$$

describes the probability for a particle with energy E to tunnel from $r < r_1$ to $r > r_2$. This formula is very useful since it can be applied to all kinds of tunneling processes. For in-

[8]The effective potential is the sum of the original potential $V(r)$ and the centrifugal term $V_{cf}(r)$ which in cylindrical coordinates has the form $V_{cf}(r) = \frac{L_z^2 - \frac{\hbar^2}{4}}{r^2}$.

stance Gamow described the alpha decay via tunneling using (A.35) [Gamow, 1928]. More recent papers show that the semiclassical treatment of particles in gravitational backgrounds makes it possible to understand Hawking radiation of Black Holes as tunneling of particles through the event horizon [Parikh and Wilczek, 2000]. Recent attempts in applying the same method to the Unruh effect (cf. [de Gill et al., 2010]) motivated the autor of the present thesis to describe the excitation of detectors in rotating frames as the tunneling of negative energy particles. But this failed due to reasons mentioned at the end of 2.3.

Table A.1 lists different values of the total[9] semiclassical transmission coefficient Γ_{WKB} (cf. (A.35)) of negative energy particles and the corresponding excitation rate (4.24) of the detector.

$\Delta E eV$	$R(\Delta E)\frac{\hbar}{eV}$	$\Gamma_{WKB}(-\Delta E)\frac{eV}{c\hbar}$	$R(\Delta E)\frac{\hbar}{eV}$	$\Gamma_{WKB}(-\Delta E)\frac{eV}{c\hbar}$
2.5	1.75×10^{-28}	1.20×10^{-25}	3.18×10^{-9}	5.95×10^{-7}
5.0	3.01×10^{-53}	4.00×10^{-50}	1.84×10^{-15}	6.52×10^{-13}
7.5	5.25×10^{-78}	1.04×10^{-74}	1.12×10^{-21}	5.82×10^{-19}
	$v = 0.1c$, $\Omega = \frac{\pi}{8}\frac{\hbar}{eV}$		$v = 0.1c$, $\Omega = \frac{\pi}{2}\frac{\hbar}{eV}$	

Table A.1.: Nummerical values of Γ_{WKB} and R.

[9]The term 'total' means that the degeneracy of the energy eigenstate has been taken into account.

B. Appendix B

B.1. Contracted Christoffel symbols

The aim of this section is to proof the identity

$$\Gamma^{\mu}{}_{\mu\lambda} = \frac{1}{\sqrt{-g}}\partial_{\lambda}\sqrt{-g}. \tag{B.1}$$

If δ denotes some derivation δg can be written as[1]

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu},$$

so that

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}.$$

Now the general form of the Christoffel symbols $\Gamma^{\mu}{}_{\nu\lambda} = \frac{1}{2}g^{\mu\alpha}\left(\partial_{\nu}g_{\alpha\lambda} + \partial_{\lambda}g_{\nu\alpha} - \partial_{\alpha}g_{\nu\lambda}\right)$ implies that the l.h.s. of (B.1) reduces to

$$\Gamma^{\mu}{}_{\mu\lambda} = \frac{1}{2}g^{\mu\alpha}\partial_{\lambda}g_{\mu\alpha}.$$

If we replace $\delta \to \partial_{\lambda}$, we find (B.1).

B.2. Klein-Gordon equation

In this section we want to give a rigorous derivation of (2.2) starting form the general form

$$\left(g^{\mu\nu}\nabla_{\mu}\nabla_{\nu} + m_0^2\right)\psi(x) = 0.$$

First we note that the wavefunction ψ is a scalar. Therefore we have $\nabla_{\mu}\psi = \partial_{\mu}\psi$, which implies that the first term gives

$$g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\psi = \nabla_{\mu}\left(g^{\mu\nu}\nabla_{\nu}\psi\right) = \nabla_{\mu}\left(g^{\mu\nu}\partial_{\nu}\psi\right).$$

The term inside the brackets is a contravariant vector $V^{\mu} = g^{\mu\nu}\partial_{\nu}\psi$, which has the covariant divergence $\nabla_{\mu}V^{\mu} = \partial_{\mu}V^{\mu} + \Gamma^{\mu}{}_{\mu\lambda}V^{\lambda}$. Combining this with (B.1) yields

$$\frac{1}{\sqrt{-g}}\partial_{\mu}\left(\sqrt{-g}g^{\mu\nu}\partial_{\nu}\psi\right) + m_0^2\psi = 0. \tag{B.2}$$

[1] We have used Jacobi's formula, see [Smirnov and Silverman, 1970] (p.165).

This form is more suitable for us, since it avoids the direct calculation of the Christoffel symbols. Now inserting (1.7) for $g^{\mu\nu}$ the first term on the l.h.s. gives us

$$\frac{1}{r}\frac{\partial}{\partial t}\left[r\frac{\partial\psi}{\partial t}\right]+\frac{1}{r}\frac{\partial}{\partial t}\left[r(-\Omega)\frac{\partial\psi}{\partial\bar{\varphi}}\right]+\frac{1}{r}\frac{\partial}{\partial\bar{\varphi}}\left[r(-\Omega)\frac{\partial\psi}{\partial t}\right]+ \\ \frac{1}{r}\frac{\partial}{\partial r}\left[-r\frac{\partial\psi}{\partial r}\right]+\frac{1}{r}\frac{\partial}{\partial\bar{\varphi}}\left[r\left(-\frac{1}{r^2}(1-\Omega^2r^2)\right)\frac{\partial\psi}{\partial\bar{\varphi}}\right]+\frac{1}{r}\frac{\partial}{\partial z}\left[-r\frac{\partial\psi}{\partial z}\right]. \quad \text{(B.3)}$$

After using the Leibniz rule and collecting some terms we obtain

$$\frac{\partial^2\psi}{\partial t^2}-2\Omega\frac{\partial^2\psi}{\partial t\partial\bar{\varphi}}+\Omega^2\frac{\partial^2\psi}{\partial\bar{\varphi}^2}-\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}-\frac{1}{r^2}\frac{\partial^2\psi}{\partial\bar{\varphi}^2}-\frac{\partial^2\psi}{\partial z^2}+m_0^2\psi=0. \quad \text{(B.4)}$$

B.3. Feynman propagator

In the following we introduce the Feynman propagator G_F of the Klein Gordon field[2]. G_F is defined as the solution of

$$(\Box+m_0^2)G_F(x)=-\delta^{(4)}(x), \quad \Box:=\partial_\mu\partial^\mu \quad \text{(B.5)}$$

with the conditions [Mukhanov and Winitzki, 2007]

$$G_F(x)\propto e^{-i\omega_{\vec{k}}x^0} \quad \text{for}\, x^0\to+\infty \quad \text{(B.6)}$$
$$G_F(x)\propto e^{i\omega_{\vec{k}}x^0} \quad \text{for}\, x^0\to-\infty. \quad \text{(B.7)}$$

(B.5) shows that G_F is a Green function of the Klein-Gordon equation in Cartesian coordinates. In order to find G_F we use the corresponding Fouier representation[3]

$$G_F(x)=\frac{1}{(2\pi)^4}\int \mathrm{d}^4k\tilde{G}_F(k)e^{-ikx}, \quad kx:=k^\mu x_\mu \quad \text{(B.8)}$$

and insert (B.8) into (B.5) to obtain

$$\frac{1}{(2\pi)^4}\int \mathrm{d}^4k\left(-(k^0)^2+\vec{k}^2+m_0^2\right)\tilde{G}_F(k)e^{-ikx}=\frac{1}{(2\pi)^4}\int \mathrm{d}^4k(-1)e^{-ikx}.$$

This implies the following algebraic equation:

$$\left((k^0)^2-\omega_{\vec{k}}^2\right)\tilde{G}_F(k)=1. \quad \text{(B.9)}$$

In the space of distributions the most general solution of this equation is given by:

$$\tilde{G}(k)=\mathcal{P}\frac{1}{(k^0)^2-\omega_{\vec{k}}^2}+\frac{C_1}{2\omega_{\vec{k}}}\delta(k^0-\omega_{\vec{k}})+\frac{C_2}{2\omega}\delta(k^0+\omega_{\vec{k}}), \quad C_{1,2}=constant. \quad \text{(B.10)}$$

[2] G_F appears often in QFT and describes the causal propagation of particles between two spacetime points.

[3] Note that G_F is a tempered distribution and therefore an element of the dual of the Schwartz space S. Consequently $\tilde{G}_F$ exists since the Fourier transform is an automorphism of S. The latter statement is the so called Fourier inversion theorem (see [Reed and Simon, 1980] for more details).

Here the symbol $\mathcal{P}\frac{1}{(k^0)^2-\omega_{\vec{k}}^2}$ indicates the principal value distribuion. Inserting (B.10) into (B.8) gives the most general Green function of the KGE

$$G(x) = \frac{1}{(2\pi)^4}\int \mathrm{d}^3k\mathcal{P}\int \mathrm{d}k^0 \frac{e^{-ikx}}{(k^0)^2-\omega_{\vec{k}}^2} + C_1\int \frac{\mathrm{d}^3k}{2\omega_{\vec{k}}}e^{-i\omega_{\vec{k}}x^0+i\vec{k}\cdot\vec{x}} + C_2\int \frac{\mathrm{d}^3k}{2\omega_{\vec{k}}}e^{i\omega_{\vec{k}}x^0-i\vec{k}\cdot\vec{x}}, \quad \text{(B.11)}$$

where the factors $(2\pi)^{-4}$ of the last two integrals have been absorbed into the constants $C_{1,2}$. The last two terms of (B.11) represent the general solution of the homogeneous KGE. The constants can be found from the boundary conditions after one has evaluated the principal value integral. We calculate this integral as follows:

- For $x^0 > 0$ we close the k^0-integration in the lower complex k^0 half-plane and subtract the contributions coming from the paths $\mathcal{C}_-$ and $\mathcal{C}_+$ as shown in Figure B.1.
- For $x^0 < 0$ we close the integration path in the upper half-plane and subtract the contributions from $\mathcal{C}_-$ and $\mathcal{C}_+$.

Note that the contributions around the poles are

$$\int_{\mathcal{C}_\pm} \mathrm{d}k^0 \frac{e^{-ik^0x^0}}{(k^0)^2-\omega_{\vec{k}}^2} = \mp i\pi\frac{e^{\mp i\omega_{\vec{k}}x^0}}{2\omega_{\vec{k}}},$$

which implies that

$$\mathcal{P}\int \mathrm{d}k^0 \frac{e^{-ikx}}{(k^0)^2-\omega_{\vec{k}}^2} = \Theta(x^0)\left(-i\pi\frac{e^{-i\omega_{\vec{k}}x^0+i\vec{k}\cdot\vec{x}}}{2\omega_{\vec{k}}} + i\pi\frac{e^{i\omega_{\vec{k}}x^0+i\vec{k}\cdot\vec{x}}}{2\omega_{\vec{k}}}\right)$$
$$\Theta(-x^0)\left(i\pi\frac{e^{-i\omega_{\vec{k}}x^0+i\vec{k}\cdot\vec{x}}}{2\omega_{\vec{k}}} - i\pi\frac{e^{i\omega_{\vec{k}}x^0+i\vec{k}\cdot\vec{x}}}{2\omega_{\vec{k}}}\right).$$

Consequently the Green function satisfies the boundary conditions (B.6) and (B.7) only if

$$C_2 = -i\pi = C_1.$$

Hence the Feynman propagator can be written as

$$G_F(x) = -\frac{i}{(2\pi)^3}\int \mathrm{d}^3k\left(\Theta(x^0)\frac{e^{-i\omega_{\vec{k}}x^0+i\vec{k}\cdot\vec{x}}}{2\omega_{\vec{k}}} + \Theta(-x^0)\frac{e^{-i\omega_{\vec{k}}x^0-i\vec{k}\cdot\vec{x}}}{2\omega_{\vec{k}}}\right), \quad \text{(B.12)}$$

where in the second term the transformation $\vec{k} \mapsto -\vec{k}$ was performed.

Note that if we replace $\mathcal{P}\int$ in (B.11) by the contour integral

$$\int_{\mathcal{C}_F} \mathrm{d}k^0 \frac{e^{-ikx}}{(k^0)^2-\omega_{\vec{k}}^2} \quad \text{(B.13)}$$

with $\mathcal{C}_F$ being the contour shown in Figure B.2 the constants $C_{1,2}$ must vanish in order to satisfy the boundary conditions (B.6) and (B.7). This choice of contour is often indicated

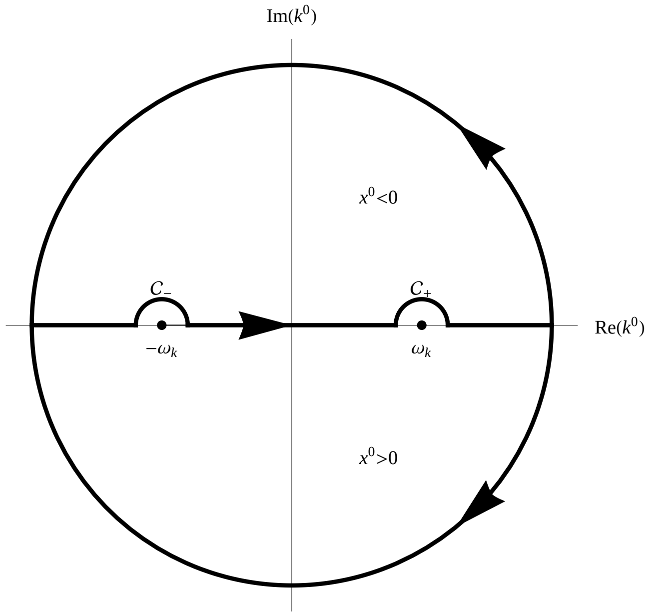

Figure B.1.: Integration contours in complex k^0-plane for evaluating the principal value integral.

by the so-called Feynman prescription

$$G_F(x) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4k \frac{e^{-ikx}}{k^2 - m_0^2 + i\epsilon} \tag{B.14}$$

with the understanding that the limit $\epsilon \to 0$ is taken after all integrations are performed.

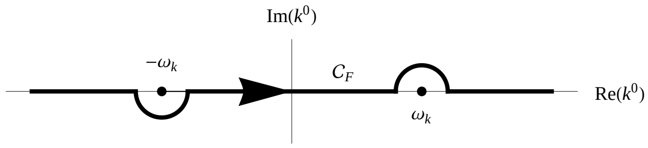

Figure B.2.: Feynman contour $\mathcal{C}_F$. The integration path must be closed either in the upper (for $x^0 < 0$) or lower (for $x^0 > 0$) half-plane.

The correct form of the Feynman propagator $G_F(x, y)$ between two spacetime points x and y is given by replacing $x \mapsto (x - y)$ in (B.14). This follows from the fact that G_F is a Lorentz scalar and that other combinations would destroy Lorentz invariance.

B.4. Wightman function of the massless scalar field

In this section we show how one obtains the result (4.16) for the massless Wightman function directly from the definition of G. From (4.7) we have for massless particles

$$G(x,y) = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3k}{2}\frac{e^{-i|\vec{k}|(x^0-y^0)+i\vec{k}\cdot(\vec{x}-\vec{y})}}{|\vec{k}|}.$$

Now we introduce spherical coordinates (r,ϑ,φ) such that $\vec{k}\cdot(\vec{x}-\vec{y}) = |\vec{k}||\vec{x}-\vec{y}|\cos\vartheta$. Then the integration measure becomes $\mathrm{d}^3k = -\mathrm{d}\varphi\mathrm{d}|\vec{k}|\mathrm{d}(\cos\vartheta)|\vec{k}|^2$ and yields

$$\begin{aligned}
&\frac{1}{(2\pi)^3}\underbrace{\int_0^{2\pi}\mathrm{d}\varphi}_{2\pi}\int_0^\infty \mathrm{d}|\vec{k}|\frac{|\vec{k}|}{2}e^{-i|\vec{k}|(x^0-y^0)}\underbrace{\int_{-1}^{1}\mathrm{d}(\cos\vartheta)e^{i|\vec{k}||\vec{x}-\vec{y}|\cos\vartheta}}_{\frac{1}{i|\vec{k}||\vec{x}-\vec{y}|}\left(e^{i|\vec{k}||\vec{x}-\vec{y}|}-e^{-i|\vec{k}||\vec{x}-\vec{y}|}\right)}\\
=&\frac{1}{(2\pi)^2}\frac{1}{|\vec{x}-\vec{y}|}\int_0^\infty \mathrm{d}|\vec{k}|e^{-i|\vec{k}|(x^0-y^0)}\sin\left(|\vec{k}||\vec{x}-\vec{y}|\right)
\end{aligned}$$

This integral does not converge in the usual sense, but converges in the distributional sense. (The field operator is actually an operator valued distribution). It may be defined as the weak limit obtained by introducing a damping factor $e^{-\epsilon|\vec{k}|}$ and removing it after the integration is done.

Hence

$$\begin{aligned}
G(x,y) &= \frac{1}{(2\pi)^2}\frac{1}{|\vec{x}-\vec{y}|}\underbrace{\int_0^\infty \mathrm{d}|\vec{k}|e^{-i|\vec{k}|(x^0-y^0-i\epsilon)}\sin\left(|\vec{k}||\vec{x}-\vec{y}|\right)}_{-\frac{|\vec{x}-\vec{y}|}{(x^0-y^0-i\epsilon)^2-|\vec{x}-\vec{y}|^2}}\\
&= -\frac{1}{4\pi^2}\frac{1}{(x^0-y^0-i\epsilon)^2-|\vec{x}-\vec{y}|^2}
\end{aligned}$$

indeed reproduces the massless limit of the massive Wightman function (4.16).

At this point we need to mention that the cutoff introduced above is not unique. There are different methods allowed, such as $e^{-\epsilon|\vec{k}|^2}$ (or replacing $\infty \to k_{max}$ and take the limit $k_{max}\to\infty$ afterwards). But all methods lead to the same result [Mukhanov and Winitzki, 2007]. The damping factor arises by adding $-i\epsilon$ to x^0-y^0. Therefore the function obtained can be regarded as the boundary value $\epsilon\to 0+$ of an analytic function of x^0-y^0 in the lower complex half-plane.

B.5. Orthogonality relation for Bessel functions

In the following we prove the orthogonality relation

$$\int\limits_0^\infty \mathrm{d}q q J_m(qr) J_m(qr') = \frac{\delta(r-r')}{r}. \tag{B.15}$$

We start form the identity

$$\delta(x-x')\delta(y-y') = \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \mathrm{d}k_1 \mathrm{d}k_2 e^{i(k_1x+k_2y)} e^{-i(k_1x'+k_2y')} \tag{B.16}$$

and introduce polar coordinates of the following type:

$$\begin{aligned} k_1 &= q\cos\theta, \quad x = r\cos\varphi, \quad x' = r'\cos\varphi' \\ k_2 &= q\sin\theta, \quad y = r\sin\varphi, \quad y' = r'\sin\varphi'. \end{aligned}$$

So the r.h.s. gives

$$\frac{1}{(2\pi)^2}\int\limits_0^{2\pi} \mathrm{d}\theta \int\limits_0^\infty \mathrm{d}q e^{iqr\cos(\theta-\varphi)} e^{-iqr'\cos(\theta-\varphi')}.$$

The Jacobi- Anger expansion $e^{i\xi\cos\phi} = \sum_{m=-\infty}^{\infty} i^m J_m(\xi) e^{im\phi}$ [Abramowitz and Stegun, 1964] yields

$$\begin{aligned} &\frac{1}{(2\pi)^2}\sum_{m=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty} i^{m-m'} \underbrace{\int\limits_0^{2\pi} \mathrm{d}\theta e^{i\theta(m-m')}}_{2\pi\delta_{mm'}} \int\limits_0^\infty \mathrm{d}q q J_m(qr) J_{m'}(qr') e^{im'\varphi' - im\varphi} \\ =&\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int\limits_0^\infty \mathrm{d}q q J_m(qr) J_m(qr') e^{im(\varphi'-\varphi)} \end{aligned}$$

Thus the identity (B.16) reads

$$\frac{\delta(r-r')}{r}\delta(\varphi-\varphi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int\limits_0^\infty \mathrm{d}q q J_m(qr) J_m(qr') e^{im(\varphi'-\varphi)}.$$

Multiplication with $e^{-im'\varphi'}$ and integration over φ' turns this into

$$\begin{aligned} \frac{\delta(r-r')}{r} e^{-im'\varphi} =& \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int\limits_0^\infty \mathrm{d}q q J_{m'}(qr) J_{m'}(qr') e^{-im\varphi} \underbrace{\int\limits_0^{2\pi} \mathrm{d}\varphi' e^{i\varphi'(m-m')}}_{2\pi\delta_{mm'}} \\ =& e^{-im'\varphi}\int\limits_0^\infty \mathrm{d}q q J_{m'}(qr) J_{m'}(qr'), \end{aligned}$$

which, after division by $e^{-im'\varphi}$, is the desired identity.

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Abstract

The present thesis examines a new method for the description of the Unruh radiation in rotating frames. The detection of particles in the Minkowski vacuum for rotating observers is to be described by means of tunneling of negative energy particles. In contrast to the well-known quantum field theoretical detector model the tunneling method provides a purely quantum mechanical approach to the subject. In order to verify this tunneling picture a comparison with the detector method was performed. Thus the tunneling probability is compared with the excitation rate of the detector. The aim was to obtain new insights into this fundamental effect which relativizes the particle concept.

First the rotating frame of reference is introduced and the corresponding free particle world lines are investigated. Then the system is quantized and the charge density and tunneling probability of negative energy particles are calculated. Subsequently the fundamentals of QFT both in inertial and rotating frames are briefly discussed. The associated vacuum definitions and their correlations are analyzed with the help of Bogoliubov transformations. In the following a simple detector model is introduced. In this context different world lines and the corresponding excitation of the detector are examined. Additionally the excitation rate of a circulating detector is compared with the tunneling probability of negative energy particles in the rotating frame. Both the exact as well as the semiclassically approximated tunneling probability are found to differ from the excitation rate of the detector. However it is shown that the excitation rate is proportional to the charge density of negative energy particles.

The transmission coefficient does not coincide with the probability of finding a particle because tunneling into a spatially finite region is considered. Therefore the normalization of the wave function is crucial and the tunneling probability differs from the detector's excitation rate.

Zusammenfassung

Diese Diplomarbeit untersucht eine neue Methode für die Beschreibung der Unruh-Strahlung in rotierenden Bezugssystemen. Die Detektion von skalaren Teilchen im Minkowski-Vakuum für rotierende Beobachter soll mit dem Tunnelvorgang von Teilchen mit negativer Energie in Zusammenhang gebracht werden. Im Gegensatz zum bekannten quantenfeldtheoretischen Detektormodell stellt das Tunneln einen rein quantenmechanischen Zugang zur vorliegenden Problemstellung dar. Um das verwendete Tunnelbild zu verifizieren wird es dem Detektormodell gegenübergestellt. Folglich wird die Tunnelwahrscheinlichkeit mit der Anregungsrate des Detektors verglichen. Dabei ist das Ziel neue Erkenntisse über diesen fundamentalen (den Teilchenbegriff relativierenden) Effekt zu erlangen.

Zunächst wird das rotierende Bezugssystem eingeführt und die zugehörigen Weltlinien von freien Teilchen untersucht. Im Anschluss wird das System quantisiert und sowohl die Ladungsdichte als auch die Tunnelwahrscheinlichkeit von Teilchen mit negativer Energie berechnet. Im dritten Kapitel werden die Grundlagen der Quantenfeldtheorie für inertiale und rotierende Beobachter kurz erläutert. Dabei werden die zugehörigen Vakuumdefinitionen und deren Zusammenhang mit Hilfe von Bogoliubov-Transformationen analysiert. Anschließend wird ein einfaches Modell eines Detektors eingeführt und dessen Anregung für verschiedene Bewegungsarten des Detektors untersucht. Insbesondere wird die Anregungsrate für eine Kreisbahn mit der Tunnelwahrscheinlichkeit von Teilchen mit negativer Energie verglichen. Dabei stellt sich heraus, dass sowohl die exakte als auch semiklassisch genäherte Tunnelwahrscheinlichkeit von der Anregungsrate des Detektors abweichen. Es wird aber die Proportionalität der Anregungsrate zur Ladungsdichte von Teilchen mit negativer Energie gezeigt.

Der Transmissionskoeffizient stimmt mit der Aufenthaltswahrscheinlichkeit nicht überein, weil die klassisch verbotene Zone räumlich begrenzt ist. Daher ist die Normierung der Wellenfunktion wesentlich und die Tunnelwahrscheinlichkeit entspricht nicht der Anregungsrate des Detektors.

Curriculum vitæ

Ich wurde am 10.01.1985 als Sohn von Juso und Mevla Kaltak in Okreč (Bosnien und Herzegovina) geboren.

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